Direct and inverse time-harmonic elastic scattering from point-like and extended obstacles

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Abstract

This paper concerns the time-harmonic direct and inverse elastic scattering by an extended rigid elastic body surrounded by a finite number of point-like obstacles. We first justify the point-interaction model for the Lamé operator within the singular perturbation approach. For a general family of pointwise-supported singular perturbations, including anisotropic and non-local interactions, we derive an explicit representation of the scattered field. In the case of isotropic and local point-interactions, our result is consistent with the ones previously obtained by Foldy's formal method as well as by the renormalization technique.

In the case of multiple scattering with pointwise and extended obstacles, we show that the scattered field consists of two parts: one is due to the diffusion by the extended scatterer and the other one is a linear combination of the interactions between the point-like obstacles and the interaction between the point-like obstacles with the extended one.

As to the inverse problem, the factorization method by Kirsch is adapted to recover simultaneously the shape of an extended elastic body and the location of point-like scatterers in the case of isotropic and local interactions. The inverse problems using only one type of elastic waves (i.e. pressure or shear waves) are also investigated and numerical examples are present to confirm the inversion schemes.

Keywords: Linear elasticity, point-like scatterers, Navier equation, Green's tensor, far field pattern.

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1 Introduction

We deal with the elastic scattering of a time-harmonic plane wave from an inhomogeneous isotropic medium in $\mathbb{R}^n$ $(n = 2, 3)$ characterized by the mass density function $\rho := \rho(x)$ and the Lamé constants $\lambda, \mu \in \mathbb{R}$ satisfying

$$\mu > 0, \quad n\lambda + 2\mu > 0. \tag{1.1}$$

It is supposed that the background medium is homogeneous, isotropic and that the inhomogeneous medium occupies a bounded domain Ω with the Lipschitz boundary $\partial\Omega$. In particular, Ω is allowed to contain a finite number of disconnected components, but its exterior $\Omega^e := \mathbb{R}^n \backslash \overline{\Omega}$ is always connected. For simplicity we assume that $\rho \equiv 1$ in $\Omega^e := \mathbb{R}^n \backslash \overline{\Omega}$. In linear elasticity, the elastic

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displacement is then governed by the time-harmonic Navier equation

$$(\Delta^* + \omega^2)\, u = 0 \quad \text{in} \quad \Omega^e, \qquad \Delta^* := \mu\Delta + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div} \tag{1.2}$$

where $\omega > 0$ denotes the angular frequency of incitation and $u = u^{in} + u^{sc}$ is the sum of the incident and scattered fields. Since the domain Ω^e is infinity in all directions $\hat{x} := x/|x| \in \mathbb{S}^{n-1} := \{|\hat{x}| = 1\}$, the scattered field u^{sc} is required to satisfy the outgoing Kupradze radiation conditions

$$\lim_{r\to\infty} r^{(n-1)/2}(\tfrac{\partial u_p}{\partial r} - ik_p u_p) = 0\,, \qquad \lim_{r\to\infty} r^{(n-1)/2}(\tfrac{\partial u_s}{\partial r} - ik_s u_s) = 0\,, \qquad r = |x|\,, \tag{1.3}$$

uniformly in all directions. Here,

$$k_p := \omega/\sqrt{\lambda + 2\mu}\,, \qquad k_s := \omega/\sqrt{\mu}\,, \tag{1.4}$$

are the compressional and shear wavenumbers of the background medium, and

$$u_p := -k_p^{-2}\mathrm{grad}\,\mathrm{div}\, u^{sc}\,, \qquad u_s = k_s^{-2}\mathrm{curlcurl}\, u^{sc}\,, \tag{1.5}$$

denote the longitudinal (compressional) and transversal (shear) parts of the scattered field in $\Omega^e \subset \mathbb{R}^3$, respectively. Note that in two dimensions the transversal (shear) part should be modified to be

$$u_s = k_s^{-2}\,\overrightarrow{\mathrm{curl}}\,\mathrm{curl}\, u^{sc}, \tag{1.6}$$

where the two curl operators in $\mathbb{R}^2$ are defined by

$$\overrightarrow{\mathrm{curl}}\, v = \partial_1 v_2 - \partial_2 v_1, \quad v = (v_1, v_2), \qquad \mathrm{curl}\, f := (\partial_2 f, -\partial_1 f).$$

It follows from the Navier equation (1.2) and the decompositions (1.5)-(1.6) that in Ω^e,

$$(\Delta + k_\alpha^2)\, u_\alpha = 0, \quad \alpha = p, s, \qquad \mathrm{div}\, u_s = 0, \qquad \mathrm{curl}\, u_p \ (\text{or } \overrightarrow{\mathrm{curl}}\, u_p) = 0.$$

The Kupradze radiation condition (1.3) leads to the P-part (longitudinal part) u_p^∞ and the S-part (transversal part) u_s^∞ of the far-field pattern of u^{sc}, given by the asymptotic behavior

$$u^{sc}(x) = \frac{\exp(ik_p|x|)}{|x|^{\frac{n-1}{2}}}\, u_p^\infty(\hat{x}) + \frac{\exp(ik_s|x|)}{|x|^{\frac{n-1}{2}}}\, u_s^\infty(\hat{x}) + \mathcal{O}(|x|^{-\frac{n+1}{2}}), \quad |x| \to +\infty, \tag{1.7}$$

where, with some normalization, u_p^∞ and u_s^∞ are the far-field patterns of u_p and u_s, respectively. In this paper, we define the far-field pattern u^∞ of the scattered field u^{sc} as the sum of u_p^∞ and u_s^∞, that is, $u^\infty := u_p^\infty + u_s^\infty$. It is well-known that u_p^∞ is normal to $\mathbb{S}^{n-1}$ and u_s^∞ is tangential to $\mathbb{S}^{n-1}$. Hence, we have the relations

$$u_p^\infty(\hat{x}) = (u^\infty(\hat{x}) \cdot \hat{x})\, \hat{x}, \quad u_s^\infty(\hat{x}) = \begin{cases} \hat{x} \times u^\infty(\hat{x}) \times \hat{x}, & \text{if} \quad n = 3, \\ (u^\infty(\hat{x}) \cdot \hat{x}^\perp)\, \hat{x}^\perp, & \text{if} \quad n = 2, \end{cases}$$

where $\hat{x}^\perp \in \mathbb{S}^{n-1}$ is perpendicular to $\hat{x}$. Note that boundary and transmission conditions should be imposed on $\partial\Omega$, relying on the physical property of Ω. It is well known that the forward scattering problem for both penetrable and impenetrable bodies admits a unique solution $u \in H^1_{loc}(D^c)$. To prove existence of solutions we refer to [27, Chapter 7.3] for the standard integral equation method applied to rigid scatterers with C^2-smooth boundaries and to a recent paper [6] using the variational approach for treating Lipschitz boundaries. Hence, the far-field pattern is uniquely determined by the incident wave (for instance, exciting frequency and direction) and the elastic body. Throughout our paper, an elastic body will be referred as a point-like scatterer if its size is much smaller than the shear wave length and the mass density has a high contrast of certain scale, as compared to the background mass density, that will be described later. It is called an extended scatterer if its size is comparable with the shear wave length. We remark that the compressional wave length is greater than the shear wave length in an isotropic and homogeneous medium. The aim of this paper is to address the following direct and inverse problems:

- Describe the Foldy approach and the point-interaction model for elastic scattering from finitely many point-like scatterers (see Section 3 for the details).
- Present a multi-scale model for elastic scattering by both point-like scatterers and an extended rigid body (Section 4).
- Recover multi-scale elastic scatterers from far-field patterns corresponding to infinitely many plane waves with all directions excited at a fixed frequency (Section 5).

In the presence of a finite number of point-like or small scatterers embedded in a homogeneous medium, it is well known that the Born approximation models the scattering effect by neglecting the wave interaction between these scatterers. Consequently, the scattered field can be represented as a weighted linear combination of point source waves emitting from each scatterer, where the weight models the scattering strength (also called scattering coefficients). Taking into account the multiple scattering, the Foldy formal approach (see [12]) assumes that the scattering coefficient of each scatterer is proportional to the external field acting on it (which is known as the Foldy assumption) and suggests to present the scattered field as a linear combination of the interactions between the point-like obstacles by solving a linear algebraic system. From the mathematical point of view, the solution to wave scattering from M point-like obstacles can be rigorously derived from the resolvent of a perturbed elliptic operator and the Krein's inversion formula of the resolvents. In fact, point perturbation operator can be regarded as the self-adjoint extension of some symmetric operator acting on appropriate Sobolev spaces. For acoustic scattering from both point-like and extended sound-soft obstacles, the point-interaction model was derived in [16] justifying the Foldy formel method and extending it to more general models including the nonlocal interactions. A closed form of the solution to such a multiscale scattering problem was obtained in [16]. Numerically, the authors of [20, 21] established an integral equation representation based on the Foldy formal approach and proposed an iterative approach for computing the unknown densities and coefficients.

The first aim of this paper is to justify the equivalence of the Foldy approach and the point interaction model for the Lamé system. The extension of our previous work [16] to the linear elasticity turns out to be non-trivial, mainly due the vectorial nature of the governing equation which models a coupling of the propagation of compressional and shear waves. Using the abstract construction of selfadjoint extensions by Posilicano [32], we model singular perturbations, of the Lamé operator, supported on a set of points, (see subsection 3.2.1). This provides a generalized boundary conditions of impedance type on this set of points. In the particular case of local and isotropic point perturbations, we retrieve the closed form of the solution obtained in [17] by the renormalization techniques arising from quantum mechanics [1]; see subsection 3.2.3. The multi-scale point-interaction model for elastic scattering from a combination of point-like and extended scatterers can be analogously formulated. In Section 4, we present a straightforward proof to the well-posedness of the resulting boundary value problem for isotropic point interactions in linear elasticity. Related to our present work, let us mention the recent contribution [7], on point-like perturbations for the two dimensional Lamé operator, where the model is stated as a selfadjoint extension of a symmetric restriction of Δ^* using boundary triplets. A factorized representation of the fields is provided in the particular case of local and isotropic perturbations.

The second aim of this paper is to investigate the inverse problem of imaging an extended rigid elastic body and a finite number of point-like scatterers,. We shall apply the factorization method [24, 25] by Kirsch to such multi-scale inverse scattering problems by using different type of elastic waves. In contrast to iterative schemes, the factorization method requires neither direct solvers nor initial guesses, and it provides a sufficient and necessary condition for characterizing the shape of the extended obstacle and positions of the point-like scatterers. Note that there is already a vast literature on inverse elastic scattering problems. The linear-sampling and factorization methods were developed in [3, 5] and [36, 9] for imaging impenetrable and penetrable scatterers. Using only one-type of elastic waves, uniqueness results for detecting extended scatterers (penetrable or impenetrable) were proved in [14, 22] and the MUSIC type algorithm [13] was applied to the detection of point-like elastic scatterers. In [18], the factorization method was adapted to recover the shape of an extended rigid body from the scattered S-waves (resp. P-waves) corresponding to all incident plane shear (resp. pressure) waves. Within the framework of this paper, we have unified the MUSIC algorithm for imaging point-like scatterers and the classical factorization scheme for recovering extended obstacles.

The remaining part of this paper is organized as follows. In Section 2, we state properties of the resolvent and outgoing Green's tensor of the Lamé operator in $\mathbb{R}^n$. Section 3 is devoted to the Foldy approach and the point-interaction model for elastic scattering by a collection of point-like scatterers. In Section 4, we present mathematical formulations for the multi-scale scattering problem and prove well-posedness of the boundary value problem. Finally, the factorization method to inverse problems together with some numerical tests are reported in Section 5.

We end up this section by introducing some notation to be used later. The spacial variables in $\mathbb{R}^n$ are denoted by $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$, where $n = 2, 3$ is the spacial dimension. Denote by $\overline{(\cdot)}$ the closure of a set or the complex conjugate of a complex number. For $a \in \mathbb{C}$, let $|a|$ denote its modulus, and for $\mathbf{a} \in \mathbb{C}^2$, let $|\mathbf{a}|$ denote its Euclidean norm. The symbol $\mathbf{a} \cdot \mathbf{b}$ stands for the inner product $a_1 b_1 + a_2 b_2$ of $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2$. Standard L^2-based scalar Sobolev spaces defined in a domain D or on a surface M are denoted by $H^s(D)$ or $H^s(M)$ for $s \in \mathbb{R}$. By $\mathsf{B}(X, Y)$ we mean the space of bounded linear operators from the space X to Y, and by $\mathbf{I}_n$ the identity matrix in $\mathbb{R}^n$.

2 Preliminaries

2.1 Properties of the resolvent of the Lamé operator in $\mathbb{R}^n$.

The quadratic form corresponding to the Lamé operator $-\Delta^*$ is given by the closed form

$$Q_0(u) := \lambda \left\| \textstyle\sum_{i=1}^n \operatorname{div} u \right\|^2_{L^2(\mathbb{R}^n)} + \tfrac{\mu}{2} \textstyle\sum_{i,j=1}^n \left\| \partial_i u_j + \partial_j u_i \right\|^2_{L^2(\mathbb{R}^n)}, \qquad u = (u_1, u_2, \cdots, u_n) \tag{2.1}$$

with $\mathsf{dom}\left(\bar{Q}_0\right) = \left(H^1(\mathbb{R}^n)\right)^n$. By (1.1), it is positive defined (see e.g. in [11, Lemma 1.1]). By [33, Theorem VIII.15], there exists a unique selfadjoint operator L_0 on $\left(L^2(\mathbb{R}^n)\right)^n$ fulfilling

$$Q_0(u) = \langle u, L_0 u \rangle_{(L^2(\mathbb{R}^n))^n}, \qquad u \in \mathsf{dom}(L_0).$$

This is the Friedrichs extension of $-\Delta^*$ and it is defined as

$$\begin{cases} \mathsf{dom}(L_0) := \left(H^2(\mathbb{R}^n)\right)^n, \\ L_0 u = -\mu \Delta u - (\lambda + \mu) \nabla (\operatorname{div} u). \end{cases} \tag{2.2}$$

Since (2.1) is positive, L_0 is positive defined and we have the resolvent set $\mathsf{res}(L_0) = \mathbb{C}\backslash[0,+\infty)$ and as $L_0 : \left(H^2(\mathbb{R}^n)\right)^n \to \left(L^2(\mathbb{R}^n)\right)^n$, it follows

$$\mathcal{K}_z := (L_0 - z)^{-1} \in \mathsf{B}\left(\left(L^2(\mathbb{R}^n)\right)^n, \left(H^2(\mathbb{R}^n)\right)^n\right), \qquad z \in \mathsf{res}(L_0). \tag{2.3}$$

Denote the Laplacian resolvent by

$$R_z := (-\Delta - z)^{-1} \in \mathsf{B}\left(H^0(\mathbb{R}^n), H^2(\mathbb{R}^n)\right), \qquad z \in \mathbb{C}\backslash[0,+\infty). \tag{2.4}$$

The *Kupradze matrix* defines the Lamé operator resolvent according to (see [28, Chp. 2])

$$\mathcal{K}_z = \frac{1}{\mu} R_{z/\mu} + \frac{1}{z}\nabla \operatorname{div}\left(R_{z/\mu} - R_{z/(\lambda+2\mu)}\right), \tag{2.5}$$

where $\mathbf{I}_n$ is the identity matrix on $\mathbb{R}^n$, while z/μ and $z/(\lambda+2\mu)$ are the rescaled energies related to compressional and shared waves. The integral kernels of R_z and $\mathcal{K}_z$ are next denoted as

$$\Phi_z(x-y) := R_z(x,y), \qquad \Gamma_z(x-y) := \mathcal{K}_z(x,y), \tag{2.6}$$

(where the identities $R_z^* = R_{\bar{z}}$ and $\mathcal{K}_z^* = \mathcal{K}_{\bar{z}}$ are taken into account). From the identity (2.5), it follows that

$$\Gamma_z(x) = \frac{1}{\mu}\Phi_{z/\mu}(x) + \frac{1}{z}\nabla \operatorname{div}\left(\Phi_{z/\mu}(x) - \Phi_{z/(\lambda+2\mu)}(x)\right). \tag{2.7}$$

We use the weighted spaces

$$H^s_\eta(\mathbb{R}^n) := \left\{u \in \mathcal{D}'(\mathbb{R}^n),\ \langle x\rangle^\eta u \in H^s(\mathbb{R}^n)\right\}, \quad s \geq 0,\ \eta \in \mathbb{R}, \tag{2.8}$$

where $\langle x\rangle^\eta := \left(1+|x|^2\right)^{\eta/2}$. The duals (w.r.t. the $L^2(\mathbb{R}^n)$ product) of (2.8) are

$$H^{-s}_{-\eta}(\mathbb{R}^n) := \left\{u \in \mathcal{D}'(\mathbb{R}^n),\ \langle x\rangle^{-\eta} u \in H^{-s}(\mathbb{R}^n)\right\}, \quad s \geq 0,\ \eta \in \mathbb{R}. \tag{2.9}$$

The Laplacian resolvent has well known mapping properties which are next recalled. At first, we recall the resolvent identity

$$R_z - R_{z_0} = (z_0 - z) R_{z_0} R_z = (z_0 - z) R_z R_{z_0}, \qquad z, z_0 \in \mathbb{C}\backslash[0,+\infty). \tag{2.10}$$

Using Fourier transform, duality and interpolation, from (2.4) follows $R_z \in \mathsf{B}\left(H^s(\mathbb{R}^n), H^{2+s}(\mathbb{R}^n)\right)$, for any $s \in \mathbb{R}$, and

$$\|R_z\|_{H^s(\mathbb{R}^n), H^{s+t}(\mathbb{R}^n)} \leq \tfrac{1}{d^{1-t/2}(z,[0,+\infty))}, \qquad t \in [0,2], \tag{2.11}$$

where $d(\cdot,[0,+\infty))$ is the distance from the set $[0,+\infty)$. According to [35, Lemma 1, p.170], one has: $R_z \in \mathsf{B}\left(L^2_\eta(\mathbb{R}^n)\right)$, for any $\eta \in \mathbb{R}$; this entails (see [29, relation (4.8)]) that $R_z \in \mathsf{B}\left(L^2_\eta(\mathbb{R}^n), H^2_\eta(\mathbb{R}^n)\right)$ and, by duality and interpolation, we get

$$R_z \in \mathsf{B}\left(H^{-s}_\eta(\mathbb{R}^n), H^{2-s}_\eta(\mathbb{R}^n)\right), \qquad \eta \in \mathbb{R}, \qquad s \in [-2,0]. \tag{2.12}$$

Since $H^s_\eta(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n)$ for $\eta > 0$, the previous mapping properties also yield

$$R_z \in \mathsf{B}\left(H^s_\eta(\mathbb{R}^n), H^{2+s}_{-\eta}(\mathbb{R}^n)\right), \qquad \eta > 0, \qquad s \in \mathbb{R}. \tag{2.13}$$

Moreover, it is well-known that $z \to R_z$ is holomorphic in $z \in \mathbb{C}\backslash [0,+\infty)$ and that a limiting absorption principle holds (see e.g. [2, Theorem 4.1], [26, Theorem 18.3]), i.e. the limits

$$R^{\pm}_{\omega^2} := \lim_{\varepsilon\to 0^+} R_{\omega^2 \pm i\varepsilon}\,, \qquad \omega > 0\,, \tag{2.14}$$

exist in $\mathsf{B}\left(L^2_\eta\left(\mathbb{R}^n\right), H^2_{-\eta}\left(\mathbb{R}^n\right)\right)$ with $\eta > 1/2$ and they satisfy

$$\left(-\Delta - \omega^2\right) R^{\pm}_{\omega^2} = 1\,. \tag{2.15}$$

This limit allows us to define the extended map

$$z \to R^{\pm}_z := \begin{cases} R_z\,, & z \in \mathbb{C}\backslash [0,+\infty)\,, \\ R^{\pm}_{\omega^2}\,, & z = \omega^2 \pm i0\,. \end{cases} \tag{2.16}$$

Using this definition, the resolvent identity extends according to

$$R^{\pm}_z - R_{z_0} = (z_0 - z)\, R_{z_0} R^{\pm}_z\,, \qquad z \in \mathbb{C}\,, \qquad z_0 \in \mathbb{C}\backslash [0,+\infty)\,. \tag{2.17}$$

Using this relation, it is easy to verify by iteration, duality and interpolation, that the limit mapping properties improve as

$$R^{\pm}_z \in \mathsf{B}\left(H^s_\eta\left(\mathbb{R}^n\right), H^{2+s}_{-\eta}\left(\mathbb{R}^n\right)\right)\,, \qquad \eta > 0\,, \qquad s \in \mathbb{R}\,. \tag{2.18}$$

The above properties of the Laplace operator extend to the Lamé operator as well. We state them in the following theorem.

Theorem 2.1. *i) Let $s \in \mathbb{R}$. For $z \in \mathbb{C}\backslash [0,+\infty)$ the map $z \to \mathcal{K}_z$ is holomorphic with values in $\mathsf{B}\left(\left(H^s\left(\mathbb{R}^n\right)\right)^n, \left(H^{s+2}\left(\mathbb{R}^n\right)\right)^n\right)$ and fulfills the estimates*

$$\left\|\mathcal{K}_z\right\|_{H^s(\mathbb{R}^n), H^{s+t}(\mathbb{R}^n)} \leq \frac{1}{d^{1-t/2}(z,[0,+\infty))}\,, \qquad t \in [0,2]\,. \tag{2.19}$$

ii) If $\eta > 1/2$, $z \to R_z$ continuously extends to the limits $z \to \omega^2 \pm i0$, $\omega > 0$ in the weaker topology of $\mathsf{B}\left(\left(H^s_\eta\left(\mathbb{R}^n\right)\right)^n, \left(H^{s+2}_{-\eta}\left(\mathbb{R}^n\right)\right)^n\right)$, i.e. the limits

$$\mathcal{K}^{\pm}_{\omega^2} := \frac{1}{\mu} R^{\pm}_{k_s^2} + \frac{1}{\omega^2} \nabla \operatorname{div}\left(R^{\pm}_{k_s^2} - R^{\pm}_{k_p^2}\right) = \lim_{\varepsilon\to 0^+} \mathcal{K}_{\omega^2 \pm i\varepsilon}\,, \tag{2.20}$$

exist in $\mathsf{B}\left(\left(H^s_\eta\left(\mathbb{R}^n\right)\right)^n, \left(H^{s+2}_{-\eta}\left(\mathbb{R}^n\right)\right)^n\right)$ with $\eta > 1/2$, $s \in \mathbb{R}$, and they satisfy

$$\left(-L_0 - \omega^2\right) \mathcal{K}^{\pm}_{\omega^2} = \mathbf{I}_n\,. \tag{2.21}$$

Proof. *i)* The mapping properties of $z \to R_z$ imply that $z \to R_{z/\mu} R_{z/(\lambda+2\mu)}$ is a $\mathsf{B}\left(\left(H^s\left(\mathbb{R}^n\right)\right)^n, \left(H^{s+4}\left(\mathbb{R}^n\right)\right)^n\right)$-valued holomorphic function in $z \in \mathbb{C}\backslash [0,+\infty)$; by (2.10), we have

$$\frac{1}{z} \nabla \operatorname{div}\left(R_{z/\mu} - R_{z/(\lambda+2\mu)}\right) = -\frac{\lambda+\mu}{\mu\left(\lambda+2\mu\right)} \nabla \operatorname{div}\left(R_{z/\mu} R_{z/(\lambda+2\mu)}\right)\,, \tag{2.22}$$

and (2.5) rephrases as

$$\mathcal{K}_z = \frac{1}{\mu} R_{z/\mu} - \frac{\lambda+\mu}{\mu\left(\lambda+2\mu\right)} \nabla \operatorname{div}\left(R_{z/\mu} R_{z/(\lambda+2\mu)}\right)\,. \tag{2.23}$$

Hence the first statement follows from the mapping properties of R_z recalled above. In particular the estimates (2.19) follow from (2.5) and (2.11). *ii)* Let $z_0, z_1, z_2 \in \mathbb{C}\setminus[0,+\infty)$; using twice (2.10) we have

$$R_{z_1} - R_{z_2} = R_{z_0}\left((z_0 - z_1) R_{z_1} - (z_0 - z_2) R_{z_2}\right) . \tag{2.24}$$

By (2.18), the r.h.s. continuously extends both to the limits $z_{j=1,2} \to \omega^2 \pm i0$ as a $\mathsf{B}\left(\left(H^s_\eta(\mathbb{R}^n)\right)^n, \left(H^{s+4}_{-\eta}(\mathbb{R}^n)\right)^n\right)$-valued map, for $\eta > 1/2$. Hence, $z \to \nabla\,\mathrm{div}\left(R_{z_s} - R_{z_p}\right)$ is continuous in $\mathbb{C}\setminus[0,+\infty)$ up to $z \to \omega^2 \pm i0$ with values in $\mathsf{B}\left(\left(H^s_\eta(\mathbb{R}^n)\right)^n, \left(H^{s+2}_{-\eta}(\mathbb{R}^n)\right)^n\right)$ and the limits are defined by

$$R^{\pm}_{k_s^2} - R^{\pm}_{k_p^2} = R_{z_0}\left(\left(z_0 - k_s^2\right) R^{\pm}_{k_s^2} - \left(z_0 - k_p^2\right) R^{\pm}_{k_p^2}\right) . \tag{2.25}$$

From (2.5), the limits (2.20) hold. Finally, the limits (2.21) follows from

$$(-L_0 - z)\,\mathcal{K}_z u = u\,, \qquad u \in \left(H^s_\eta(\mathbb{R}^n)\right)^n . \tag{2.26}$$

□

The Green kernels of the limiting operators $R^{\pm}_{k^2}$, expressed by the limits

$$\Phi^{\pm}_{k^2} := \lim_{\varepsilon\to 0^+} \Phi_{k^2 \pm i\varepsilon}\,, \tag{2.27}$$

are radiating solutions of the Helmoholtz equation in $\mathbb{R}\setminus\{0\}$, i.e. these satisfy the radiation conditions

$$\lim_{r\to 0} r^{(n-1)/2}\left(\partial_r \mp ik\right)\Phi^{\pm}_{k^2}(x) = 0\,, \quad r = |x|. \tag{2.28}$$

From (2.20), the corresponding limit kernels of $\mathcal{K}^{\pm}_{\omega^2}$ are given by

$$\Gamma^{\pm}_{\omega^2} := \frac{1}{\mu}\Phi^{\pm}_{k_s} + \frac{1}{\omega^2}\nabla\,\mathrm{div}\left(\Phi^{\pm}_{k_s} - \Phi^{\pm}_{k_p}\right) , \tag{2.29}$$

where we recall that k_p and k_s are the compressional and shear wavenumbers, respectively. Following [28], these are solutions of the Lamé stationary equation $\left(L_0 - \omega^2\right)\Gamma^{\pm}_{\omega^2} = 0$ in $\mathbb{R}\setminus\{0\}$ and, for $\omega \neq 0$, fulfill the Kupradze radiation conditions

$$\lim_{r\to 0} r^{(n-1)/2}\left(\partial_r \mp i\,k_p\right)\nabla\,\mathrm{div}\,\Gamma^{\pm}_{\omega^2} = 0\,, \tag{2.30}$$

$$\lim_{r\to 0} r^{(n-1)/2}\left(\partial_r \mp i\,k_s\right)\nabla\times\nabla\times\Gamma^{\pm}_{\omega^2} = 0\,. \tag{2.31}$$

2.2 Outgoing Green's tensor in 2D

We recall some known properties of the integral kernels Φ_z and Γ_z for $n = 2$; more details can be found in [28], [19] and references therein. If $n = 2$, the integral kernel of R_z is given by

$$\Phi_z(x - y) := \frac{i}{4} H^{(1)}_0\left(\zeta\,|x - y|\right) , \qquad \zeta \in \mathbb{C}^+ : \zeta^2 = z \in \mathbb{C}\setminus[0,+\infty) , \tag{2.32}$$

where $H^{(1)}_0$ is the Henkel function of the first kind and of order zero. By (2.7) and (2.32), the $\mathcal{C}^\infty\left(\mathbb{R}^{2,2}\right)$-valued map $z \to \Gamma_z(x)$, introduced in (2.7), is holomorphic in $z \in \mathbb{C}\setminus[0,+\infty)$ and continuously extends to the limits $z \to \omega^2 \pm i0$, $k \in \mathbb{R}$.; in particular, by [19, eq. (11)], the limit

$$\Gamma_0(x) := \lim_{z\to 0}\Gamma_z(x) = \frac{1}{4\pi}\left[-\frac{\lambda + 3\mu}{\mu(\lambda + 2\mu)}\ln|x|\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\lambda + \mu}{\mu(\lambda + 2\mu)}\frac{1}{|x|^2}\begin{pmatrix}x_1^2 & x_1 x_2\\ x_1 x_2 & x_2^2\end{pmatrix}\right] , \tag{2.33}$$

pointwise holds in $\mathbb{R}^2\backslash\{0\}$. From (2.33) we get

$$\det\left(\Gamma_0(x)\right)=\frac{\ln|x|\,(\lambda+3\mu)}{\left(4\pi\mu\,(\lambda+2\mu)\right)^2}\left((\lambda+\mu)-(\lambda+3\mu)\ln|x|\right). \tag{2.34}$$

Hence the inverse matrix $\Gamma_0^{-1}(x)$ exists in $\mathbb{R}^2\backslash\left\{0,|x|=e^{\frac{\lambda+\mu}{\lambda+3\mu}}\right\}$ where it writes as

$$\begin{aligned}\Gamma_0^{-1}(x) &= \frac{\left(4\pi\mu\,(\lambda+2\mu)\right)^2}{\ln|x|\,(\lambda+3\mu)\left((\lambda+\mu)-(\lambda+3\mu)\ln|x|\right)}\\ &\quad\times\begin{pmatrix}(\lambda+3\mu)\ln|x|-(\lambda+\mu)\frac{x_2^2}{|x|^2} & -(\lambda+\mu)\frac{x_1x_2}{|x|^2}\\ -(\lambda+\mu)\frac{x_1x_2}{|x|^2} & (\lambda+3\mu)\ln|x|-(\lambda+\mu)\frac{x_1^2}{|x|^2}\end{pmatrix}.\end{aligned} \tag{2.35}$$

It follows

$$\left|\Gamma_0^{-1}(x)\right|_{\mathbb{R}^{2,2}}=\mathcal{O}\left(1/\ln|x|\right), \qquad \text{as } x\to 0\,. \tag{2.36}$$

Let

$$\tilde{\Gamma}_z(x):=\Gamma_z(x)-\Gamma_0(x)\,; \tag{2.37}$$

by [19, Lemma 2.1] it results

$$\begin{cases}\chi_z\,\mathbf{I}_2:=\lim_{|x|\to 0}\tilde{\Gamma}_z(x)\,,\\[2ex] \chi_z=\chi_z(\lambda,\mu)=-\frac{1}{4\pi}\left[\frac{\lambda+3\mu}{\mu(\lambda+2\mu)}\left(\ln\frac{\sqrt{z}}{2}+C-i\frac{\pi}{2}\right)+\frac{\lambda+\mu}{\mu(\lambda+2\mu)}-\frac{1}{2}\left(\frac{\ln\mu}{\mu}+\frac{\ln(\lambda+2\mu)}{\lambda+2\mu}\right)\right],\end{cases} \tag{2.38}$$

where C is the Euler constant and $\sqrt{z}$ is defined with $\operatorname{Im}\sqrt{z}>0$. From (2.38) we get

$$\left|\Gamma_0^{-1}(x)\,\tilde{\Gamma}_z(x)\right|_{\mathbb{R}^{2,2}}=\mathcal{O}\left(1/\ln|x|\right), \qquad \text{as } x\to 0\,. \tag{2.39}$$

By (2.36)-(2.39) we get

$$\Gamma_0^{-1}(x)\,\Gamma_z(y)=\Gamma_0^{-1}(x)\left(\Gamma_0(y)+\tilde{\Gamma}_z(y)\right)=\begin{cases}\mathbf{I}_2+\mathcal{O}\left(1/\ln|x|\right), & \text{as } x\to 0 \text{ if } x=y\,,\\[2ex] o\left(1/\ln|x|\right), & \text{as } x\to 0 \text{ if } x\neq y\,,\end{cases} \tag{2.40}$$

holding in the $\mathbb{R}^{2,2}$-norm sense.

2.3 Outgoing Green's tensor in 3D

If $n=3$, the integral kernel of R_z is explicitly given by

$$\Phi_z(x-y)=\frac{e^{i\zeta|x-y|}}{4\pi\,|x-y|}, \qquad \zeta\in\mathbb{C}^+:\zeta^2=z\in\mathbb{C}\backslash[0,+\infty)\,. \tag{2.41}$$

By (2.7) and (2.41), the $\mathcal{C}^\infty\left(\mathbb{R}^{3,3}\right)$-valued map $z\to\Gamma_z(x)$ is holomorphic in $z\in\mathbb{C}\backslash[0,+\infty)$ and continuously extends to the limits $z\to k\pm i0$, $k\in\mathbb{R}$.; by ([19, eq. 48]), the limit

$$\left(\Gamma_0(x)\right)_{j,\ell}:=\lim_{z\to 0}\left(\Gamma_z(x)\right)_{j,\ell}=\frac{\lambda+3\mu}{8\pi\mu\,(\lambda+2\mu)}\frac{\delta_{j,\ell}}{|x|}+\frac{\lambda+\mu}{8\pi\mu\,(\lambda+2\mu)}\frac{x_jx_\ell}{|x|^3}, \tag{2.42}$$

pointwise holds in $\mathbb{R}^3 \setminus \{0\}$. From

$$\det\left(\frac{8\pi\mu\,(\lambda+2\mu)\,|x|^3}{\lambda+\mu}\Gamma_0\,(x)\right)=\left(\left(\frac{2\mu}{\lambda+\mu}\right)^2+3\frac{2\mu}{\lambda+\mu}+2\right)|x|^2>0\quad\text{in }\mathbb{R}^3\setminus\{0\}\;. \tag{2.43}$$

This allows to define the inverse tensor $\Gamma_0^{-1}\,(x)$ which writes as

$$\left(\Gamma_0^{-1}\,(x)\right)_{j,\ell}=q\,(\lambda,\mu)\,|x|\left(2\left(1+\tfrac{\mu}{\lambda+\mu}\right)|x|^2\,\delta_{j,\ell}-x_jx_\ell\right)\,,\qquad q\,(\lambda,\mu):=4\pi\mu\tfrac{\lambda+\mu}{\lambda+3\mu}\,. \tag{2.44}$$

It follows

$$\left|\Gamma_0^{-1}\,(x)\right|_{\mathbb{R}^{3,3}}=o\,(|x|)\;,\qquad\text{as }x\to 0\,. \tag{2.45}$$

As before we denote

$$\tilde{\Gamma}_z\,(x):=\Gamma_z\,(x)-\Gamma_0\,(x)\;. \tag{2.46}$$

By ([19, eq. 49]) results

$$\chi_z\,\mathbf{I}_3:=\lim_{x\to 0}\tilde{\Gamma}_z\,(x)=i\sqrt{z}\frac{2\lambda+5\mu}{12\pi\mu\,(\lambda+2\mu)}\,\mathbf{I}_3\,, \tag{2.47}$$

and from (2.45) we get

$$\left|\Gamma_0^{-1}\,(x)\,\tilde{\Gamma}_\zeta\,(x)\right|_{\mathbb{R}^{3,3}}=o\,(|x|)\;,\qquad\text{as }x\to 0\,. \tag{2.48}$$

It follows

$$\Gamma_0^{-1}\,(x)\,\Gamma_\zeta\,(y)=\Gamma_0^{-1}\,(x)\left(\Gamma_0\,(y)+\tilde{\Gamma}_\zeta\,(y)\right)=\begin{cases}\mathbf{I}_3+o\,(|x|)\;, & \text{as }x\to 0\text{ if }x=y\,,\\ \\ o\,(|x|)\;, & \text{as }x\to 0\text{ if }x\neq y\,,\end{cases} \tag{2.49}$$

holding in the $\mathbb{R}^{3,3}$-norm sense.

3 Elastic scattering by a collection of point-like obstacles

In this section we consider the time-harmonic elastic scattering by N point-like scatterers located at $y^{(k)}, k=1,\cdots,N$ in $\mathbb{R}^n$ ($n=2,3$). The set of these point-like scatterers will be denoted by $Y:=\{y^{(k)}:k=1,\cdots,N\}$. Physically, such small obstacles are related to highly concentrated inhomogeneous elastic medium (i.e. the mass density in our case) with sufficiently small diameters compared to the wave-length of incidence. In other words we shall suppose that

$$\Omega=\bigcup_{k=1}^N D_k\,,\quad \rho|_{D_k}\approx(\mathrm{diam}(D_k))^{-n}\ \ \text{and}\quad \tfrac{\omega}{2\pi}\mathrm{diam}(D_k)\ll 1\,,\qquad k=1,\cdots,N\,.$$

Note that the wave-length for compressional and shear waves are defined via

$$\lambda_p:=\frac{2\pi}{k_p}=\frac{2\pi}{\omega}\sqrt{\lambda+2\mu},\quad \lambda_s:=\frac{2\pi}{k_s}=\frac{2\pi}{\omega}\sqrt{\mu} \tag{3.1}$$

respectively. From mathematical point of view, the presence of these point-like obstacles corresponds to a formal Delta-like perturbation of the density function

$$\rho(x)-1=\sum_{k=1}^N a_k\,\delta(x-y^{(k)})\,, \tag{3.2}$$

where $a_k\in\mathbb{C}$ is the scattering strength (coupling constant) attached to the scatter located at $y^{(k)}$. We write $a=(a_1,a_2,\cdots,a_N)\in\mathbb{C}^N$ and denote by $\mathbf{I}_N$ the identity matrix in $\mathbb{C}^{N\times N}$.

3.1 Foldy approach

A formal solution to the scattering problem (1.2),(1.3) and (3.2) is given by

$$u(x) = u^{in}(x) + \sum_{m=1}^{N} a_m \, \Gamma_{\omega^2}(x, y^{(m)}) \, u(y^{(m)}), \quad x \neq y^{(m)}, \qquad m = 1, 2, \cdots, N, \tag{3.3}$$

where $\Gamma_{\omega^2}(x, y)$ is the fundamental tensor of the Lamé operator. To determine the value of u at $y^{(k)}$ on the right hand side of (3.3), the Foldy approach, originated in acoustic scattering from many particles [12, 37, 31], suggests solving the linear algebraic system

$$u(y^{(k)}) = u^{in}(y^{(k)}) + \sum_{m=1, m\neq j}^{N} a_m \, \Gamma_{\omega^2}(y^{(k)}, y^{(m)}) \, u(y^{(m)}), \quad k = 1, 2, \cdots, N. \tag{3.4}$$

In fact, the Foldy system (3.4) follows from taking the limits $x \to y^{(k)}$ in (3.3) and removing the singular term (i.e., when $m = k$) in the sum. It has been shown in ([8]) that the algebraic system is uniquely solvable except for a discrete set of frequencies and some particular distribution of the points at $y^{(k)}$. Inserting the solution of (3.4) into (3.3) we obtain an explicit representation of the solution of our scattering problem in the form of

$$u(x) = u^{in} + \sum_{m,k=1}^{N} \Gamma_{\omega^2}(x, y^{(k)}) \; [\Pi_{\omega^2}^{-1}]_{m,k} \; u^{in}(y^{(m)}), \tag{3.5}$$

where $[\Pi_{\omega^2}^{-1}]_{m,k}$ denotes the (m, k)-th entry of the inverse of the block-matrix Π_{ω^2} given by

$$\Pi_{\omega^2} := \begin{pmatrix} \mathbf{I}_n & -a_2 \, \Gamma_{\omega^2}(y^{(1)}, y^{(2)}) & \cdots & -a_N \, \Gamma_{\omega^2}(y^{(1)}, y^{(N)}) \\ -a_1 \, \Gamma_{\omega^2}(y^{(2)}, y^{(1)}) & \mathbf{I}_n & \cdots & -a_N \, \Gamma_{\omega^2}(y^{(2)}, y^{(N)}) \\ \vdots & \vdots & \ddots & \vdots \\ -a_N \, \Gamma_{\omega^2}(y^{(N)}, y^{(1)}) & -a_2 \, \Gamma_{\omega^2}(y^{(2)}, y^{(N)}) & \cdots & \mathbf{I}_n \end{pmatrix}.$$

A rigorous justification of the Foldy system (3.5) was carried out in [17] by applying the *renormalization techniques* arising from quantum mechanics for describing the point interaction of finitely many particles. Replacing the scattering coefficients a_k by parameter dependent functions $ak(\epsilon)$ that decay in a suitable way as $\epsilon \to 0$, one can show via Weinstein-Aronszajn determinant formula that the resolvent of the a family of ϵ-dependent delta perturbations of the Lamé operator converges in Agmon's weighted spaces. The resolvent of the limiting operator leads to the same expression of u as in (3.5) with

$$a_k = b_k - \chi_{\omega^2}, \quad b_k \in \mathbb{C}, \qquad k = 1, \cdots, N,$$

where $\chi_{\omega^2} \in \mathbb{C}$ are the dimension-dependent constants given by (2.38) and (2.47) with $z = \omega^2$. These quantities corresponds to the the normalizing constants introduced in [19, relations (12) and (49)].

3.2 Point-interaction of the Lamé operator

In this study, the scattering effect due to the presence of point-like obstacles is modeled as elastic point interactions. Within this model, the scattering problem will be regarded as singular perturbations supported on points.

We next consider a collection of n points $Y = \{y^{(k)}\}_{k=1}^{N} \subset \mathbb{R}^n$; by the Sobolev imbedding $\mathcal{C}^1 \hookrightarrow H^2_\eta(\mathbb{R}^n)$ holding for $n \leq 3$, $\eta \in \mathbb{R}$, the auxiliary map

$$\gamma : \left(H^2_\eta(\mathbb{R}^n)\right)^n \to \mathbb{C}^{n,N}, \quad (\gamma u)_{j,k} := u_j\left(y^{(k)}\right), \ j = 1,..n, \ k = 1,..N, \tag{3.6}$$

is continuous and surjective. From the identity

$$\langle \gamma\varphi, m\rangle_{\mathbb{R}^n} = \langle \varphi, \gamma^* m\rangle_{(H^2(\mathbb{R}^n))^n,(H^{-2}(\mathbb{R}^n))^n}, \quad \varphi \in \left(H^2_\eta(\mathbb{R}^n)\right)^n, \tag{3.7}$$

follows that

$$\gamma^* \in \mathsf{B}\left(\mathbb{C}^{n,N}, \left(H^{-2}_Y(\mathbb{R}^n)\right)^n\right) \qquad \left((\gamma^* X)(x)\right)_j := \textstyle\sum_{k=1}^N X_{j,k}\, \delta\left(x - y^{(k)}\right), \tag{3.8}$$

i,e.: $\mathsf{ran}(\gamma^*)$ is formed by H^{-2} vector-valued delta distributions supported on Y. The singular perturbations of L_0 supported on Y are defined by the selfadjoint extensions of the symmetric restriction $L_0 \restriction \ker(\gamma)$. These models are next defined following the approach developped in [30].

Lemma 3.1. *Let*

$$G_z := \mathcal{K}_z \gamma^*, \qquad z \in \mathbb{C}\backslash [0, +\infty). \tag{3.9}$$

i) The map $z \to G_z$, $z \in \mathbb{C}\backslash[0,+\infty)$, is analytic $\mathsf{B}\left(\mathbb{C}^{n,N}, \left(L^2_\eta(\mathbb{R}^n)\right)^n\right)$-valued for all $\eta \in \mathbb{R}$ and, for $\eta > 1/2$, continuosly extends to

$$G_z^\pm := \lim\nolimits_{\varepsilon\to 0^+} G_{z\pm i\varepsilon} \in \mathsf{B}\left(\mathbb{C}^{n,N}, \left(L^2_{-\eta}(\mathbb{R}^n)\right)^n\right), \qquad \eta > 1/2. \tag{3.10}$$

ii) There exists $c > 0$ (possibly depending on z and η) such that

$$\|G_z X\|_{\left(L^2_\eta(\mathbb{R}^n)\right)^n} > c\,\|X\|_{\mathbb{C}^{n,N}}, \quad \eta \in \mathbb{R}. \tag{3.11}$$

iii) For $z, z_0 \in \mathbb{C}\backslash[0,+\infty)$ and $\lambda > 0$ the identities

$$G_z - G_{z_0} = (z_0 - z)\,\mathcal{K}_{z_0} G_z, \qquad G_z^\pm - G_{z_0} = (z_0 - z)\,\mathcal{K}_{z_0} G_{z_0}^\pm, \tag{3.12}$$

hold and

$$\begin{aligned} &(G_z - G_{z_0}) \in \mathsf{B}\left(\mathbb{C}^{n,N}, \left(H^2_\eta(\mathbb{R}^n)\right)^n\right), \quad \eta \in \mathbb{R}, \\ &\left(G_z^\pm - G_{z_0}\right) \in \mathsf{B}\left(\mathbb{C}^{n,N}, \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n\right), \quad \eta > 1/2. \end{aligned} \tag{3.13}$$

iv) For $X \in \mathbb{C}^{n,N}$ it results

$$\begin{aligned} (G_z X)_\ell(x) &= \textstyle\sum_{k=1}^N \sum_{j=1}^n \left(\Gamma_z\left(x - y^{(k)}\right)\right)_{\ell,j} X_{j,k}, \\ \left(G_z^\pm X\right)_\ell(x) &= \textstyle\sum_{k=1}^N \sum_{j=1}^n \left(\Gamma_z^\pm\left(x - y^{(k)}\right)\right)_{\ell,j} X_{j,k}, \end{aligned} \tag{3.14}$$

where Γ_z denotes the resolvent Green kernels for the Lamé operator. Moreover, the outgoing/ingoing Kupradze radiation condition (2.30)/(2.31) hold for $G^\pm_{\omega^2}$.

Proof. *i)* The first point follows from (3.8) and Theorem 2.1. *ii)* By the surjectivity of the trace γ, $G_z^* = \gamma\mathcal{K}_{\bar z}$ is surjective; hence by the closed range theorem G_z has closed range and (3.11) follows from [23, Theorem VI.5.2]. *iii)* (3.12) and (3.13) follows from the first resolvent identity

$$\mathcal{K}_z = \mathcal{K}_{z_0} + (z_0 - z)\,\mathcal{K}_z \mathcal{K}_{z_0}, \qquad z, z_0 \in \mathbb{C}\backslash[0, +\infty), \tag{3.15}$$

and the mapping properties of $\mathcal{K}_z$. *iv)* Let $X \in \mathbb{C}^{n,N}$; by (3.8), (3.9) we get

$$(G_z X)_\ell (x) = \left(\mathcal{K}_z \sum_{k=1}^{N} X_{j,k}\, \delta\left(\cdot - y^{(k)}\right)\right)_\ell (x) = \sum_{k=1}^{N} \sum_{j=1}^{n} \left(\Gamma_z \left(x - y^{(k)}\right)\right)_{\ell,j} X_{j,k}\,, \tag{3.16}$$

$$\left(G_z^{\pm} X\right)_\ell (x) = \left(\mathcal{K}_z^{\pm} \sum_{k=1}^{N} X_{j,k}\, \delta\left(\cdot - y^{(k)}\right)\right)_\ell (x) = \sum_{k=1}^{N} \sum_{j=1}^{n} \left(\Gamma_z^{\pm} \left(x - y^{(k)}\right)\right)_{\ell,j} X_{j,k}\,. \tag{3.17}$$

Finally, the outgoing/ingoing Kupradze radiation condition follows from (3.14) and (2.30)-(2.31). □

For $z \in \mathbb{C}\backslash [0, +\infty)$ it results (see [30, eq. (2.15)])

$$\begin{aligned} &\mathsf{dom}\left((L_0 \restriction \ker(\gamma))^*\right) \\ &= \left\{u \in \left(L^2(\mathbb{R}^n)\right)^n\,,\ u = u_0 + G_z X\,,\ u_0 \in \left(H^2(\mathbb{R}^n)\right)^n\,,\ X \in \mathbb{C}^{n,N}\right\}\,, \end{aligned} \tag{3.18}$$

$$\left((L_0 \restriction \ker(\gamma))^* - z\right) u = (L_0 - z)\, u_0\,. \tag{3.19}$$

This representation holds for any $z \in \mathbb{C}\backslash [0, +\infty)$ and the decomposition provided in (3.18) is unique. The action of G_z on $\mathbb{C}^{n,N}$ provides a representation of the defect spaces $\ker\left((L_0 \restriction \ker(\gamma))^* - z\right)$. Namely

$$\ker\left((L_0 \restriction \ker(\gamma))^* - z\right) = \mathsf{ran}\,(G_z)\,. \tag{3.20}$$

Assumption Assume an open set $\mathbb{C}\backslash\mathbb{R} \subseteq Z_\Lambda \subseteq \mathbb{C}\backslash [0, +\infty)$ and a family $\Lambda_z \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$, $z \in Z_\Lambda$, such that

$$i)\ \Lambda_z^* = \Lambda_{\bar{z}} \qquad ii)\ \Lambda_w - \Lambda_z = (z - w)\, \Lambda_w\, (G_{\bar{w}})^*\, G_z \Lambda_z\,. \tag{3.21}$$

Following [30, Theorem 2.4] we have:

Theorem 3.2. *Let $\Lambda := \{\Lambda_z\,,\ z \in Z_\Lambda\}$ be a family of $\mathsf{B}\left(\mathbb{C}^{n,N}\right)$ maps fulfilling the conditions (3.21). Then*

$$\mathcal{K}_z^\Lambda := \mathcal{K}_z + G_z \Lambda_z\, (G_{\bar{z}})^*\,, \qquad z \in Z_\Lambda\,, \tag{3.22}$$

is the resolvent of a selfadjoint extension L_Λ of $L_0 \restriction \ker(\gamma)$.

For each $z \in Z_\Lambda$, the identity (3.22) yields

$$\mathsf{dom}\,(L_\Lambda) = \left\{u = u_0 + G_z \Lambda_z \gamma u_0\,,\ u_0 \in \left(H^2(\mathbb{R}^n)\right)^n\right\}\,. \tag{3.23}$$

Since $L_0 \subset L_\Lambda \subset (L_0 \restriction \ker(\tau))^*$, we identify: $L_\Lambda := (L_0 \restriction \ker(\tau))^* \restriction \mathsf{dom}\,(A_\Lambda)$; hence

$$L_\Lambda u = (-\mu\Delta - (\lambda + \mu)\, \nabla\, \mathrm{div})\, u \quad \text{in } \mathbb{R}^n\backslash Y \tag{3.24}$$

and from (3.20) follows

$$L_\Lambda u = L_0 u_0 - z G_z \Lambda_z \gamma u_0\,, \qquad u = u_0 + G_z \Lambda_z \gamma u_0\,,\ u_0 \in \left(H^2(\mathbb{R}^n)\right)^n\,. \tag{3.25}$$

3.2.1 Boundary conditions models

For $\theta \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ we introduce

$$\Lambda_z(\theta) := \left(\theta + \gamma\left(G_{-1} - G_z\right)\right)^{-1} . \tag{3.26}$$

Lemma 3.3. *Let $\Lambda_z(\theta)$ be given by (3.26) with $\theta \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ selfadjoint. Then there exists a (possibliy empty) discrete set $S_{\Lambda(\theta)} \subset \mathbb{R}_-$ such that $z \to \Lambda_z(\theta)$ defines an analytic family of $\mathsf{B}\left(\mathbb{C}^{n,N}\right)$-tensors in $\mathbb{C}\backslash\left\{[0,+\infty) \cup S_{\Lambda(\theta)}\right\}$. For each $z \in \mathbb{C}\backslash\left\{[0,+\infty) \cup S_{\Lambda(\theta)}\right\}$ the Assumption 3.2 holds. The limits $\Lambda^{\pm}_{\omega^2}(\theta) := \lim_{\varepsilon\to 0^+} \Lambda_{\omega^2 \pm i\varepsilon}(\theta)$ exist in $\mathsf{B}\left(\mathbb{C}^{n,N}\right)$ for a.a. $\omega > 0$, with the possible exception of a discrete subset and coincide with*

$$\Lambda^{\pm}_{\omega^2}(\theta) = \left(\theta + \gamma\left(G_{-1} - G^{\pm}_{\omega^2}\right)\right)^{-1} . \tag{3.27}$$

Proof. Let consider the direct mapping: $\left(\Lambda_z(\theta)\right)^{-1} := \left(\theta + \gamma\left(G_{-1} - G_z\right)\right)$; by $(i)-(iii)$ of Lemma 3.1 results $\left(\Lambda_z(\theta)\right)^{-1} \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ and $\left(\left(\Lambda_z(\theta)\right)^{-1}\right)^* = \left(\Lambda_{\bar{z}}(\theta)\right)^{-1}$. By

$$\left\langle Y, \left(\Lambda_z(\theta)\right)^{-1} X\right\rangle_{\mathbb{C}^{n,N}} = \left\langle \left(\Lambda_{\bar{z}}(\theta)\right)^{-1} Y, X\right\rangle_{\mathbb{C}^{n,N}} , \tag{3.28}$$

follows: $\ker\left(\Lambda_{\bar{z}}(\theta)\right)^{-1} = \left(\mathsf{ran}\left(\left(\Lambda_z(\theta)\right)^{-1}\right)\right)^{\perp}$; from (3.11) follows

$$\left|\left\langle X, \left(\Lambda_{\bar{z}}(\theta)\right)^{-1} X\right\rangle_{\mathbb{C}^{n,N}}\right| \geq \left|\mathrm{Im}\left\langle X, \left(\Lambda_{\bar{z}}(\theta)\right)^{-1} X\right\rangle_{\mathbb{C}^{n,N}}\right| = 2\left|\mathrm{Im}\, z\right| \left\|G_z X\right\|^2_{(L^2(\mathbb{R}^n))^n} > 2c\left|\mathrm{Im}\, z\right| \left\|X\right\|_{\mathbb{C}^{n,N}} . \tag{3.29}$$

Then, $\ker\left(\Lambda_{\bar{z}}(\theta)\right)^{-1} = \{0\}$ and $\left(\Lambda_z(\theta)\right)^{-1}$ is bijective in $\mathbb{C}\backslash\mathbb{R}$: this allows to define $\Lambda_z(\theta) \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ for each $z \in \mathbb{C}\backslash\mathbb{R}$; since $z \to \left(\Lambda_z(\theta)\right)^{-1}$ is analytic (tensor-valued) in $\mathbb{C}\backslash[0,+\infty)$, by the properties of analytic functions in finite dimensional spaces the inverse tensor-valued map $z \to \left(\Lambda_z(\theta)\right)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and extends to a meromorphic function in $\mathbb{C}\backslash[0,+\infty)$. Furthermore, it continuosly extends to the limits $z \to \omega^2 \pm i0$ for a.a. $k > 0$ and from

$$\begin{aligned} \mathbf{I}_{n\times N} &= \left(\lim_{\varepsilon\to 0^+} \Lambda_{\omega^2 \pm i\varepsilon}(\theta)\right)\left(\theta + \gamma\left(G_{-1} - G^{\pm}_{\omega^2}\right)\right) \\ &= \left(\theta + \gamma\left(G_{-1} - G^{\pm}_{\omega^2}\right)\right)\left(\lim_{\varepsilon\to 0^+} \Lambda_{\omega^2 \pm i\varepsilon}(\theta)\right) , \end{aligned} \tag{3.30}$$

the identity (3.27) follows. Let $z \in \mathbb{C}\backslash\left\{[0,+\infty) \cup D_{\Lambda(\theta)}\right\}$; from: $\left(\theta + \gamma\left(G_{-1} - G_{\bar{z}}\right)\right) = \left(\theta + \gamma\left(G_{-1} - G_z\right)\right)^*$ we have

$$\Lambda_{\bar{z}}(\theta) = \left(\theta + \gamma\left(G_{-1} - G_{\bar{z}}\right)\right)^{-1} = \left(\left(\theta + \gamma\left(G_{-1} - G_z\right)\right)^*\right)^{-1} = \Lambda^*_z(\theta) . \tag{3.31}$$

Finally, from (3.12) follows

$$\left(\theta + \gamma\left(G_{-1} - G_w\right)\right) - \left(\theta + \gamma\left(G_{-1} - G_z\right)\right) = \gamma\left(G_z - G_w\right) = (w - z)\, G^*_{\bar{w}} G_z , \tag{3.32}$$

which implies

$$\Lambda_w(\theta) - \Lambda_z(\theta) = (z - w)\, \Lambda_w(\theta)\, G^*_{\bar{w}} G_z \Lambda_z(\theta) . \tag{3.33}$$

□

The construction of Theorem 3.2 is next implemented with the family of tensors given in (3.26).

Lemma 3.4. *Let $\Lambda_z(\theta)$ be given by (3.26) with $\theta \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ selfadjoint. For $z \in \mathbb{C}\backslash\left\{[0,+\infty) \cup D_{\Lambda(\theta)}\right\}$, the domain (3.23) rephrases as*

$$\mathsf{dom}\left(L_{\Lambda(\theta)}\right) = \left\{u = u_0 + G_{-1}X\,,\ u_0 \in \left(H^2\left(\mathbb{R}^n\right)\right)^n\,,\ X \in \mathbb{C}^{n,N} : \gamma u_0 = \theta X\right\}\,. \tag{3.34}$$

Proof. By (3.23), $u \in \mathsf{dom}\left(L_{\Lambda(\theta)}\right)$ implies: $u = u_0 + G_z\Lambda_z\gamma u_0$, with $u_0 \in \left(H^2\left(\mathbb{R}^n\right)\right)^n$. By (3.13) we have

$$u = \tilde{u}_0 + G_{-1}\Lambda_z(\theta)\gamma u_0\,, \qquad \tilde{u}_0 = u_0 - \left(G_{-1} - G_z\right)\Lambda_z(\theta)\gamma u_0 \in \left(H^2\left(\mathbb{R}^n\right)\right)^n\,, \tag{3.35}$$

with

$$\gamma\tilde{u}_0 = \gamma u_0 - \gamma\left(G_{-1} - G_z\right)\Lambda_z(\theta)\gamma u_0 = \gamma u_0 + \theta\Lambda_z(\theta)\gamma u_0 - \left(\Lambda_z(\theta)\right)^{-1}\Lambda_z(\theta)\gamma u_0 = \theta\Lambda_z(\theta)\gamma u_0\,. \tag{3.36}$$

Setting $X = \Lambda_z(\theta)\gamma u_0$, we get: $\gamma\tilde{u}_0 = \theta X$. Hence u belongs to the set (3.34). Let $u = u_0 + G_{-1}X$, be defined with $u_0 \in \left(H^2\left(\mathbb{R}^n\right)\right)^n$ and $X \in \mathbb{C}^{n,N}$ such that: $\gamma u_0 = \theta X$. Then

$$u = \tilde{u}_0 + G_zX\,, \qquad \tilde{u}_0 = u_0 + \left(G_{-1} - G_z\right)X \in \left(H^2\left(\mathbb{R}^n\right)\right)^n\,. \tag{3.37}$$

Setting: $\tilde{X} = \left(\Lambda_z(\theta)\right)^{-1}X$ we get $u = \tilde{u}_0 + G_{-1}\Lambda_z(\theta)\tilde{X}$ with

$$\tilde{X} = \left(\Lambda_z(\theta)\right)^{-1}X = \theta X + \gamma\left(G_{-1} - G_z\right)X = \gamma u_0 + \gamma\left(G_{-1} - G_z\right)X = \gamma\tilde{u}_0\,. \tag{3.38}$$

□

With the notation introduced in Sections 2.2 and 2.3, $\Gamma_z(x-y)$ denotes the resolvent Green kernel for the Lamé operator and $\Gamma_0(x-y)$ its $z \to 0$ limit: these are $\mathbb{C}^{n,n}$-valued tensors field and, according to (2.35) and (2.44), the inverse matrix $\Gamma_0^{-1}(x)$ pointwise exists for $x \neq 0$. Let us define the maps $\tau_{j=1,2} : \mathsf{dom}\left(\left(L_0 \upharpoonright \ker(\gamma)\right)^*\right) \to \mathbb{C}^{n,N}$

$$\left(\tau_1 u\right)_{j,k} := \lim_{x \to y^{(k)}} \textstyle\sum_{\ell=1}^{n}\left(\Gamma_0^{-1}\left(x - y^{(k)}\right)\right)_{j,\ell} u_\ell(x)\,, \tag{3.39}$$

$$\left(\tau_2 u\right)_{j,k} := \lim_{x \to y^{(k)}} \left(u_j(x) - \textstyle\sum_{\ell=1}^{n}\left(\Gamma_0\left(x - y^{(k)}\right)\right)_{j,\ell}\left(\tau_1 u\right)_{\ell,k}\right) \tag{3.40}$$

for all $j = 1,..n\,,\ k = 1,..N$. Let us remark that $\tau_{j=1,2}$ extend to $\left(H^2_\eta\left(\mathbb{R}^n\right)\right)^n$ functions for any $\eta \in \mathbb{R}$. These maps allows to represent the operator's domain in terms of boundary conditions. For $z \in \mathbb{C}\backslash[0,+\infty)$ we introduce $\Xi_z \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$

$$\left(\Xi_z X\right)_{j,k} := \textstyle\sum_{k'=1,\ k' \neq k}^{N}\left(\Gamma_z\left(y^{(k)} - y^{(k')}\right)X\right)_{j,k}\,, \qquad X \in \mathbb{C}^{n,N}\,. \tag{3.41}$$

Proposition 3.5. *Let $\Lambda_z(\theta)$ be given by (3.26) with $\theta \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ selfadjoint. The domain (3.23) rephrases as*

$$\mathsf{dom}\left(L_{\Lambda(\theta)}\right) = \left\{u \in \mathsf{dom}\left(\left(L_0 \upharpoonright \ker(\gamma)\right)^*\right) : \tau_2 u = \left(\Xi_{-1} + \theta + \chi_{-1}\,\mathbf{I}_{n\times N}\right)\tau_1 u\right\}\,, \tag{3.42}$$

where Ξ_{-1} and χ_{-1} are defined by (3.41) and (2.38), (2.47) for $z = -1$.

Proof. From (2.36) and (2.45), follows

$$\tau_1 u_0 = 0\,, \qquad u_0 \in \left(H^2_\eta\left(\mathbb{R}^n\right)\right)^n\,, \qquad \eta \in \mathbb{R}\,. \tag{3.43}$$

Moreover

$$(\tau_1 G_z X)_{j,k} = \lim_{x\to y^{(k)}} \sum_{\ell=1}^{n} \left(\Gamma_0^{-1}\left(x-y^{(k)}\right)\right)_{j,\ell} \left(\sum_{k'=1}^{N}\sum_{j'=1}^{n} \left(\Gamma_z\left(x-y^{(k')}\right)\right)_{\ell,j'} X_{j',k'}\right)$$

$$= \lim_{x\to y^{(k)}} \sum_{k'=1}^{N}\sum_{j'=1}^{n} \left(\sum_{\ell=1}^{n} \left(\Gamma_0^{-1}\left(x-y^{(k)}\right)\right)_{j,\ell} \left(\Gamma_z\left(x-y^{(k')}\right)\right)_{\ell,j'}\right) X_{j',k'}$$

$$= \lim_{x\to y^{(k)}} \sum_{k'=1}^{N}\sum_{j'=1}^{n} \left(\Gamma_0^{-1}\left(x-y^{(k)}\right)\Gamma_z\left(x-y^{(k')}\right)\right)_{j,j'} X_{j',k'}\,. \tag{3.44}$$

From (2.40) and (2.49) we obtain

$$(\tau_1 G_z X)_{j,k} = \lim_{x\to y^{(k)}} \sum_{j'=1}^{n} \left(\Gamma_0^{-1}\left(x-y^{(k)}\right)\Gamma_z\left(x-y^{(k)}\right)\right)_{j,j'} X_{j',k}$$

$$+ \lim_{x\to y^{(k)}} \sum_{\substack{k'=1\\ k'\neq k}}^{N}\sum_{j'=1}^{n} \left(\Gamma_0^{-1}\left(x-y^{(k)}\right)\Gamma_z\left(x-y^{(k')}\right)\right)_{j,j'} X_{j',k'} = X_{j,k}\,. \tag{3.45}$$

It follows

$$\tau_1 G_z = \mathbf{I}_{n\times N}\,, \qquad z\in\mathbb{C}\backslash\left[0,+\infty\right)\,. \tag{3.46}$$

Let $u\in\mathsf{dom}\left(\left(L_0\restriction\ker\left(\gamma\right)\right)^*\right)$; by (3.18) there exist $u_0\in\left(H^2\left(\mathbb{R}^n\right)\right)^n$ and $X\in\mathbb{C}^{n,N}$ such that: $u=u_0+G_{-1}X$. By (3.43) and (3.46) we have

$$\tau_1\left(u_0+G_{-1}X\right)=X\,, \qquad u_0\in\left(H^2_\eta\left(\mathbb{R}^n\right)\right)^n\,,\ X\in\mathbb{C}^{n,N}\,,\ \eta\in\mathbb{R}\,. \tag{3.47}$$

By (3.40), (3.14),and (3.47) we have

$$\left(\tau_2\left(u_0+G_{-1}X\right)\right)_{j,k}$$

$$= \left(u_0\right)_j\left(x_k\right) + \lim_{x\to y^{(k)}}\left(\sum_{k'=1}^{N}\sum_{\ell=1}^{n}\left(\Gamma_{-1}\left(x-y^{(k')}\right)\right)_{j.\ell} X_{\ell,k'} - \sum_{\ell=1}^{n}\left(\Gamma_0\left(x-y^{(k)}\right)\right)_{j,\ell} X_{\ell,k}\right)$$

$$= \left(u_0\right)_j\left(x_k\right) + \sum_{\substack{k'=1\\ k'\neq k}}^{N}\sum_{\ell=1}^{n}\left(\Gamma_{-1}\left(y^{(k)}-y^{(k')}\right)\right)_{j,\ell} X_{\ell,k'}$$
$$+ \lim_{x\to y^{(k)}}\sum_{\ell=1}^{n}\left(\Gamma_{-1}\left(x-y^{(k)}\right)-\Gamma_0\left(x-y^{(k)}\right)\right)_{j,\ell} X_{\ell,k}\,. \tag{3.48}$$

By (2.37)-(2.38) and (2.46)-(2.47) we have

$$\lim_{x\to y^{(k)}}\sum_{\ell=1}^{n}\left(\lim_{|x|\to 0}\tilde{\Gamma}_{-1}\left(x\right)\right)_{j,\ell} X_{\ell,k} = \chi_{-1}\,X_{j,k}\,, \tag{3.49}$$

where χ_{-1} is given by (2.38) and (2.47) for $z=-1$. From (3.41) we have

$$\Xi_{-1}\in\mathsf{B}\left(\mathbb{C}^{N,n}\right)\,, \qquad \left(\Xi_{-1}\,X\right)_{j,k} = \sum_{k'=1,\ k'\neq k}^{N}\left(\Gamma_{-1}\left(y^{(k)}-y^{(k')}\right)X\right)_{j,k}\,. \tag{3.50}$$

Using (3.49)-(3.50) allows to rephrase (3.48) as

$$\tau_2\left(u_0+G_{-1}X\right)=\gamma u_0+\left(\Xi_{-1}+\chi_{-1}\right)X\,, \qquad u_0\in\left(H^2_\eta\left(\mathbb{R}^n\right)\right)^n\,,\ X\in\mathbb{C}^{n,N}\,,\ \eta\in\mathbb{R}\,. \tag{3.51}$$

In wiev of (3.18), Lemma 3.4, (3.47) and (3.51), the domain (3.23) identifies with (3.42). $\square$

3.2.2 The diffusion problem

For $\Lambda_z(\theta)$ given by (3.26) with $\theta \in \mathsf{B}(\mathbb{C}^{n,N})$ selfadjoint, we introduce an extended operator $\tilde{L}_{\Lambda(\theta)}$: $\mathsf{dom}\left(\tilde{L}_{\Lambda(\theta)}\right) \to L^2_{-\eta}(\mathbb{R}^3)$ defined for $\eta > 1/2$ and $z \in \mathbb{C}\backslash\mathbb{R}$ by

$$\begin{cases} \mathsf{dom}\left(\tilde{L}_{\Lambda(\theta)}\right) = \left\{u \in \left(L^2_{-\eta}(\mathbb{R}^n)\right)^n,\ u = u_0 + G_z\Lambda_z(\theta)\gamma u_0,\ u_0 \in \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n\right\}, \\ \tilde{L}_{\Lambda(\theta)}u = (-\mu\Delta - (\lambda+\mu)\nabla\operatorname{div})\,u, \qquad \text{in } \mathbb{R}^n\backslash Y \end{cases} \tag{3.52}$$

This model can be characterized in terms of the boundary conditions introduced in Proposition 3.5.

Lemma 3.6. *Let $\Lambda_z(\theta)$ and $\tilde{L}_{\Lambda(\theta)}$ be defined according to (3.26) and (3.52) with $\theta \in \mathsf{B}(\mathbb{C}^{n,N})$ selfadjoint. Then, for each k such that the limits (3.27) exist,* $\mathsf{dom}\left(\tilde{L}_{\Lambda(\theta)}\right)$ *admits the representations*

$$\mathsf{dom}\left(\tilde{L}_{\Lambda(\theta)}\right) = \left\{u \in \left(L^2_{-\eta}(\mathbb{R}^n)\right)^n,\ u = u_0 + G^{\pm}_{\omega^2}\Lambda^{\pm}_{\omega^2}(\theta)\gamma u_0,\ u_0 \in \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n\right\}, \tag{3.53}$$

and the boundary conditions hold

$$\tau_2 u = (\Xi_{-1} + \theta + \chi_{-1})\,\tau_1 u, \qquad u \in \mathsf{dom}\left(\tilde{L}_{\Lambda(\theta)}\right). \tag{3.54}$$

Proof. By (3.12)-(3.13), $u = u_0 + G_z\Lambda_z(\theta)\gamma u_0$ identifies with

$$u = \tilde{u}_0 + G^{\pm}_{\omega^2}\Lambda_z(\theta)\gamma u_0, \qquad \tilde{u}_0 = u_0 + \left(\omega^2 - z\right)\mathcal{K}^{\pm}_{\omega^2}G_z\Lambda_z(\theta)\gamma u_0 \in \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n. \tag{3.55}$$

Using (3.21) we get

$$\Lambda_z(\theta)\gamma u_0 = \Lambda^{\pm}_{\omega^2}(\theta)\left(1 + \left(\omega^2 - z\right)\left(G^{\mp}_{\omega^2}\right)^* G_z\Lambda_z(\theta)\right)\gamma u_0 = \Lambda^{\pm}_{\omega^2}(\theta)\gamma\tilde{u}_0, \tag{3.56}$$

from which it follows

$$u = \tilde{u}_0 + G^{\pm}_{\omega^2}\Lambda^{\pm}_{\omega^2}(\theta)\tilde{u}_0, \qquad \tilde{u}_0 = u_0 + \left(\omega^2 - z\right)\mathcal{K}^{\pm}_{\omega^2}G_z\Lambda_z(\theta)\gamma u_0 \in \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n. \tag{3.57}$$

This shows that

$$\mathsf{dom}\left(\tilde{L}_{\Lambda(\theta)}\right) \subseteq \left\{u \in \left(L^2_{-\eta}(\mathbb{R}^n)\right)^n,\ u = u_0 + G^{\pm}_{\omega^2}\Lambda^{\pm}_{\omega^2}(\theta)\gamma u_0,\ u_0 \in \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n\right\}. \tag{3.58}$$

Using again (3.12)-(3.13) and (3.21) a similar argument leads to the opposite inclusion. Proceeding as in Lemma 3.4, we get

$$\mathsf{dom}\left(\tilde{L}_{\Lambda(\theta)}\right) = \left\{u = u_0 + G_{-1}X,\ u_0 \in \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n,\ X \in \mathbb{C}^{n,N} : \gamma u_0 = \theta X\right\}. \tag{3.59}$$

and the boundary conditions (3.54) follows from (3.47) and (3.51). □

For $\omega > 0$ such that the limits (3.27) exist, the generalized eigenfunctions of energy ω^2 are the solutions of the problem

$$\left(\tilde{L}_{\Lambda(\theta)} - \omega^2\right)u = 0, \qquad u \in \mathsf{dom}\left(\tilde{L}_{\Lambda(\theta)}\right). \tag{3.60}$$

Lemma 3.7. *For $\omega > 0$ such that the limits (3.27) exist, the solutions of (3.60) express as*

$$u = u_0 + G^{\pm}_{\omega^2} \Lambda^{\pm}_{\omega^2}(\theta)\, \gamma u_0 \,, \tag{3.61}$$

where $u_0 \in \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n$ is a generalized eigenfunction of L_0.

Proof. Let $u \in \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n$ be a generalized eigenfunction of L_0, and define u according to (3.61). Then, by (3.52), we get

$$\left(\tilde{L}_{\Lambda(\theta)} - \omega^2\right)\left(u_0 + G^{\pm}_{\omega^2} \Lambda^{\pm}_{\omega^2}(\theta)\, \gamma u_0\right) = -\left(\mu\Delta + (\lambda + \mu)\nabla \operatorname{div} + \omega^2\right) u_0 = 0 \,. \tag{3.62}$$

$\square$

Let u^{sc} denote the stationary diffusion of an incoming wave $u^{in} := u_0 \in \left(H^2_{-\eta}(\mathbb{R}^n)\right)^n$ (a generalized eigenfunction of L_0); we have: $u = u^{in} + u^{sc} \in \mathsf{dom}\left(\tilde{L}_{\Lambda(\theta)}\right)$ and by (3.54), u^{sc} solves the boundary condition problem

$$\begin{cases} \left(\mu\Delta + (\lambda + \mu)\nabla \operatorname{div} + \omega^2\right) u^{sc} = 0 \,, & \text{in } \mathbb{R}^n \backslash Y \\ \tau_2\left(u^{sc} + u^{in}\right) = \left(\Xi_{-1} + \theta + \chi_{-1}\right) \tau_1\left(u^{sc} + u^{in}\right) \,, & \end{cases} \tag{3.63}$$

and fulfills the Kupradze outgoing radiation conditions (1.3). By Lemma 3.7, this problem admits the unique solution

$$u^{sc} = G^{+}_{\omega^2} \Lambda^{+}_{\omega^2}(\theta)\, \gamma u^{in} \,. \tag{3.64}$$

Remark 3.8. *The construction presented above provides a large class of point perturbation models, including anisotropic and non-local interactions. From the physical point of view, anisotropy refers to different scattering properties depending on the direction, while non-locality refers to a coupling between different point scatterers. While the applications considered in this work focus on isotropic local perturbations, it is worth noticing that the scattering theory presented here holds in a much more general framework.*

3.2.3 Modelling local isotropic point perturbations

Let us start with a characterization of the class of models we are interested in. The notion of isotropy corresponds to the fact that the effect of the perturbation is independent from the direction, while locality excludes the possible couplings between the points. These properties and the domain representation in (3.42) motivate the next definition.

Definition 3.9. *Let $\theta \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ be selfadjoint and let $\{e_{j,k}\}$ denote the standard basis in $\mathbb{C}^{n,N}$. Let Ξ_{-1} and χ_{-1} be defined by (3.41), (2.38) and (2.47) for $z = -1$. We say that the operator $L_{\Lambda(\theta)}$, with $\Lambda_z(\theta)$ given in (3.26), is an* isotropic *perturbation when*

$$\left\langle e_{j,k}, \left(\theta + \Xi_{-1} + \chi_{-1}\, \mathbf{I}_{n\times N}\right)\left(e_{j,k'}\right)\right\rangle_{\mathbb{C}^{n,N}} = \left\langle e_{j',k}, \left(\theta + \Xi_{-1} + \chi_{-1}\, \mathbf{I}_{n\times N}\right)\left(e_{j',k'}\right)\right\rangle_{\mathbb{C}^{n,N}} \,, \tag{3.65}$$

for all $j, j' = 1, ..n$ and $k, k' = 1, ..N$. The perturbation is local *if*

$$\left\langle e_{j,k}, \left(\theta + \Xi_{-1} + \chi_{-1}\, \mathbf{I}_{n\times N}\right)\left(e_{j',k'}\right)\right\rangle_{\mathbb{C}^{n,N}} = 0 \,, \qquad \forall\, j, j' = 1, ..n \text{ and } k \neq k' \,. \tag{3.66}$$

As an example, consider $\alpha \in \mathbb{R}^{N,N}$ and define $\alpha \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ as

$$\alpha\left(M\right) := M\alpha\,, \qquad M \in \mathbb{C}^{n,N}\,. \tag{3.67}$$

The operator $\theta\left(\alpha\right) := \alpha - \Xi_{-1} - \chi_{-1}\,\mathbf{I}_{n\times N} \in \mathsf{B}\left(\mathbb{C}^{N,n}\right)$ is selfadjoint. This choice enters in the scheme of the Proposition 3.5 and a corresponding perturbed Lamé operator $L_{\Lambda(\theta(\alpha))}$ is defined with the boundary conditions (3.23). Is easy to see in this framework that the scalar products in (3.65) are independent from the direction. Indeed, we have

$$\left\langle e_{j,k}, \left(\theta\left(\alpha\right) + \Xi_{-1} + \chi_{-1}\,\mathbf{I}_{n\times N}\right)\left(e_{j,k'}\right)\right\rangle_{\mathbb{C}^{n,N}} = \left\langle e_{j,k}, \alpha\left(e_{j,k'}\right)\right\rangle_{\mathbb{C}^{n,N}} = \alpha_{k'k}\,, \qquad \forall\, j = 1,..n\,. \tag{3.68}$$

Hence the perturbation $L_{\Lambda(\theta(\alpha))}$ is isotropic. The condition of locality is satisfied when

$$\left\langle e_{j,k}, \left(\theta\left(\alpha\right) + \Xi_{-1} + \chi_{-1}\,\mathbf{I}_{n\times N}\right)\left(e_{j',k'}\right)\right\rangle_{\mathbb{C}^{n,N}} = \left\langle e_{j,k}, \alpha\left(e_{j',k'}\right)\right\rangle_{\mathbb{C}^{n,N}} = \alpha_{k'k} = 0\,, \qquad \forall\, k \neq k'\,,$$

which corresponds to the choice of a diagonal α. We resume these properties in the following lemma.

Lemma 3.10. *Let $\alpha \in \mathbb{R}^{N}$ and define $\alpha \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ as*

$$\alpha\left(M\right) := M\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \in \mathbb{C}^{n,N}\,, \qquad M \in \mathbb{C}^{n,N}\,. \tag{3.69}$$

The perturbation $L_\alpha := L_{\Lambda(\theta(\alpha))}$, defined by $\theta\left(\alpha\right) := \alpha - \Xi_{-1} - \chi_{-1}\,\mathbf{I}_{n\times N}$, is isotropic and local.

In what follows, we consider an isotropic and local perturbation L_α with $\alpha \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ selfadjoint given by (3.69). The boundary conditions in (see Proposition 3.5) are

$$\tau_2 u = \alpha\left(\tau_1 u\right)\,, \qquad u \in \mathsf{dom}\left(\left(L_0 \restriction \ker\left(\gamma\right)\right)^*\right)\,, \tag{3.70}$$

and componentwise reads as

$$\left(\tau_2 u\right)_{j,k} = \alpha_k\,\left(\tau_1 u\right)_{j,k}\,, \qquad j = 1,..n\,,\ k = 1,..N\,. \tag{3.71}$$

With the notation introduced in Proposition 3.5, we define

$$\Lambda_z^\alpha := \left(\alpha - \Xi_{-1} - \chi_{-1}\,\mathbf{I}_{n\times N} + \gamma\left(G_{-1} - G_z\right)\right)^{-1}\,. \tag{3.72}$$

Setting

$$\mathsf{dom}\left(L_\alpha\right) = \left\{u \in \mathsf{dom}\left(\left(L_0 \restriction \ker\left(\gamma\right)\right)^*\right) : \tau_2 u = \alpha\,\left(\tau_1 u\right)\right\}\,, \tag{3.73}$$

and using the construction of Theorem 3.2 and Proposition 3.5, we have

$$L_\alpha := \left(L_0 \restriction \ker\left(\gamma\right)\right)^* \restriction \mathsf{dom}\left(L_\alpha\right)\,. \tag{3.74}$$

Rephrasing in this framework the result of the previous sections, by Lemma 3.3 the limit maps $\Lambda_{\omega^2}^{\alpha,\pm} \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ exist for $\omega^2 \in (0,+\infty)\backslash S_\alpha$ where $S_\alpha \subset (0,+\infty)$ is a discrete subset. Under this condition, the stationary diffusion problem of an incoming wave u^{in} reads as

$$\begin{cases} \left(\mu\Delta + \left(\lambda+\mu\right)\nabla\operatorname{div} + \omega^2\right)u^{sc} = 0\,, & \text{in } \mathbb{R}^n\backslash Y \\ \tau_2\left(u^{sc} + u^{in}\right) = \alpha\,\tau_1\left(u^{sc} + u^{in}\right)\,, & \\ \text{The outgoing radiation conditions in (1.3) hold.} & \end{cases} \tag{3.75}$$

Lemma 3.11. *Let $\alpha \in \mathsf{B}\left(\mathbb{C}^{N,n}\right)$ be defined by (3.69), $\omega^2 \in (0,+\infty)\backslash S_\alpha$ and $u^{in} \in \left(H^2_{-\eta}\left(\mathbb{R}^n\right)\right)^n$ be a generalized eigenfunction of energy ω^2 of L_0. The unique solution of (3.75) is given by*

$$
\begin{aligned}
&(u^{sc})_\ell\,(x) = \frac{1}{\mu}\textstyle\sum_{k=1}^N \Phi^+_{k_s}\left(x-y^{(k)}\right)\left(\Lambda^{\alpha,+}_{\omega^2}\gamma u^{in}\right)_{\ell,k}(x)\\
&+\frac{1}{\omega^2}\textstyle\sum_{k=1}^N \partial_\ell\left(\sum_{j=1}^n \partial_j\left(\Phi^+_{k_s}\left(x-y^{(k)}\right)-\Phi^+_{k_p}\left(x-y^{(k)}\right)\right)\left(\Lambda^{\alpha,+}_{\omega^2}\gamma u^{in}\right)_{j,k}(x)\right). \qquad (3.76)
\end{aligned}
$$

Proof. The representation (3.76) follows from (3.64) by taking into account (2.20) and (3.14). □

Remark 3.12. *From the identity*

$$
\begin{aligned}
\gamma\left(G_{-1}-G_z\right) &= \lim_{x\to 0}\left(\Gamma_{-1}(x)-\Gamma_z(x)\right)\mathbf{I}_n + \Xi_{-1}-\Xi_z\\
&= \lim_{x\to 0}\left(\Gamma_{-1}(x)-\Gamma_0(x)\right)\mathbf{I}_n - \lim_{x\to 0}\left(\Gamma_z(x)-\Gamma_z(x)\right)\mathbf{I}_n + \Xi_{-1}-\Xi_z\,,
\end{aligned}
$$

and the limits (2.38), (2.47), follows

$$
\gamma\left(G_{-1}-G_z\right) = \chi_{-1}\mathbf{I}_{n\times N} - \chi_z\,\mathbf{I}_{n\times N} + \Xi_{-1}-\Xi_z\,. \qquad (3.77)
$$

Then (3.72) rephrases as

$$
\Lambda^\alpha_z := \left(\alpha - \chi_z\,\mathbf{I}_{n\times N} - \Xi_z\right)^{-1}\,, \qquad (3.78)
$$

and the corresponding limits $\Lambda^{\alpha,+}_{\omega^2}$ corresponds to the inverse of

$$
\left(\Lambda^\alpha_{\omega^2}\right)^{-1} := \left(\alpha - \chi_{\omega^2}\mathbf{I}_{n\times N} - \Xi_{\omega^2}\right)
$$

$$
= \begin{pmatrix}
\left(\alpha_1-\chi_{\omega^2}\right)\mathbf{I}_n & -\Gamma_{\omega^2}\left(y^{(1)}-y^{(2)}\right) & \cdots & -\Gamma_{\omega^2}\left(y^{(1)}-y^{(N)}\right)\\
-\Gamma_{\omega^2}\left(y^{(2)}-y^{(1)}\right) & \left(\alpha_2-\chi_{\omega^2}\right)\mathbf{I}_n & \cdots & -\Gamma_{\omega^2}\left(y^{(2)}-y^{(N)}\right)\\
\vdots & & & \vdots\\
-\Gamma_{\omega^2}\left(y^{(N)}-y^{(1)}\right) & \cdots & -\Gamma_{\omega^2}\left(y^{(N)}-y^{(N-1)}\right) & \left(\alpha_n-\chi_{\omega^2}\right)\mathbf{I}_n
\end{pmatrix}. \qquad (3.79)
$$

The representation (3.79) is consistent with the ones provided in [19, Sec. II and III] for isotropic point perturbations models using the regularization approach.

4 Elastic scattering by point-like and extended obstacles

In this section, we consider the scattering of elastic incident waves from a multi-scale scatterer $\Omega = D \cup Y$, where D is an extended obstacle and $Y := \{y^{(j)} : j = 1, 2, \cdots, N\} \subset \mathbb{R}^n \backslash \overline{D}$ represents a set of finitely many point-like elastic scatterers. For simplicity we assume that the extended scatterer D is a rigid elastic body. However, our augments can be easily adpted to other penetrable or impenetrable extended scatterers.

In what follows we focus on the case where Ω is formed by point scatterer and an extended sound-soft (i.e. Dirichlet) obstacle. We will next denote with $u(x) = u^{in}(x) + u^{sc}(x)$ the total field corresponding to the multiple scattering of an incident wave u^{in} on the point scatterers in Y and the Dirichlet extended obstacle D, while $u_D = u^{in} + u^{sc}_D \in (H^1_{loc}(\mathbb{R}^n\backslash\overline{\Omega}))^n$ is the total field in the absence of the point obstacles, i.e.: u_D^{sc} is the unique Kupradze outgoing radiation-solution to the boundary value problem

$$\left(\Delta^* + \omega^2\right) u_D^{sc} = 0 \quad \text{in} \quad \mathbb{R}^n \backslash \overline{D}, \qquad u_D^{sc} = -u^{in} \quad \text{on} \quad \partial D\,. \tag{4.1}$$

The fundamental solution for the Navier equation in $\mathbb{R}^n \backslash \overline{D}$ with Dirichlet boundary condition on ∂D, next denoted with $\Gamma_D(x,y)$, is a $\mathbb{C}^{n,n}$ tensor field defined by: $\Gamma_D(x,y) := \Gamma_D^{sc}(x,y) + \Gamma_{\omega^2}(x,y)$ where $\Gamma_{\omega^2}(x,y)$ ($y \in \mathbb{R}^n \backslash \overline{D}$) is the free space Green's tensor to the Navier equation, while $\Gamma_D^{sc}(x,y)$ is the unique solution to

$$\left(\Delta^* + \omega^2\right) \Gamma_D^{sc}(\cdot, y) = 0 \quad \text{in} \quad \mathbb{R}^n \backslash \overline{D}, \qquad \Gamma_D^{sc}(\cdot, y) = -\Gamma_{\omega^2}(\cdot, y) \quad \text{on} \quad \partial D\,. \tag{4.2}$$

Motivated by the "impedance"-type boundary condition (3.75) for modelling local and isotropic point perturbations established in Section 3.2, we assume that the boundary conditions (3.71) hold for the total field, i.e.

$$(\tau_2 u)_{j,k} = (\tau_1 u)_{j,k} \alpha_k\,, \qquad \alpha \in \mathbb{C}^N\,, \qquad j = 1,..n\,,\ k = 1,..N\,. \tag{4.3}$$

According to the definition of the mappings $\tau_{\ell=1,2}$ (see (3.39) and (3.40)) we have the following asymptotic behavior

$$u_j(x) = \textstyle\sum_{j'=1,..n} \left(\Gamma_0(x, y^{(k)})\right)_{j,j'} (\tau_1 u)_{j',k} + (\tau_2 u)_{j,k} + O(|x - y^{(k)}|) \qquad \text{as} \quad x \to y^{(k)}. \tag{4.4}$$

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in \mathbb{C}^N$ and define as before $\alpha \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$ according to (3.69); to describe the solution of (1.2), (1.3) and (4.3), we introduce the modified tensor $\Lambda_{\omega^2}^{\alpha,D} \in \mathsf{B}\left(\mathbb{C}^{n,N}\right)$, whose inverse is defined by the $\mathbb{C}^{n,n}$ matrix-block entries

$$\left(\Lambda_{\omega^2}^{\alpha,D}\right)_{k,k'}^{-1} = \begin{cases} -\Gamma_D(y^{(k)}, y^{(k')})\,, & k \neq k'\,, \\ (\alpha_k - \chi_{\omega^2})\,\mathbf{I}_n\,, & k = k'\,, \end{cases} \qquad k, k' = 1,..N\,, \tag{4.5}$$

with the constant χ_{ω^2} given by (2.38) and (2.47) for $z = \omega^2$. We define the set

$$S_\alpha^D := \{\omega > 0 : \det \left(\Lambda_{\omega^2}^{\alpha,D}\right)^{-1} = 0\}\,, \tag{4.6}$$

and address the exterior stationary diffusion problem of an incoming wave u^{in}

$$\begin{cases} \left(\Delta^* + \omega^2\right) u^{sc} = 0\,, \qquad \text{in } \mathbb{R}^n \backslash \Omega\,, \\ u^{sc} = -u^{in}\,, \qquad \text{on } \partial D\,, \\ \tau_2\left(u^{sc} + u^{in}\right) = \alpha\,\tau_1\left(u^{sc} + u^{in}\right)\,, \\ \text{The outgoing radiation conditions in (1.3) hold.} \end{cases} \tag{4.7}$$

We remark that, when $\alpha \in \mathbb{R}^N$, this multiscale scattering model can be justified either using the Foldy's formal approach, the renormalization technique or the point-interaction approach, following the same arguments presented in Section 3.2. In particular, the solution can be derived as in Lemmas 3.7 and 3.11 by replacing the Green's tensor Γ_{ω^2} with Γ_D and by replacing the incident wave u^{in} with u_D^{in}, respectively (see (4.8) below).

Theorem 4.1. *Assume that $\omega \notin S_\alpha^D$ and $Im\,\alpha_j \leq 0$ for all $k = 1, \cdots, N$. Then, the boundary value problem problem (4.7) admits a unique solution in $H^1_{loc}(\mathbb{R}^3 \backslash \overline{\Omega})$, which represents as*

$$u(x) = u_D(x) + \sum_{k,k'=1,..N} \Gamma_D(x, y^{(k)}) \left(\Lambda_{\omega^2}^{\alpha,D}\right)_{k,k'} u_D\left(y^{(k')}\right)\,, \qquad x \in \mathbb{R}^3 \backslash \overline{\Omega}\,, \tag{4.8}$$

where $u_D = u^{in} + u_D^{sc}$ is the total field in the absence of the point-like obstacles and the sums over the space-idices are hidden.

The above theorem shows that the scattered field caused by $D \cup Y$ consists of two parts: one is due to the diffusion by the extended scatterer (i.e., u_D) and the other one is a linear combination of the interactions between the point-like obstacles and the interaction between the point-like obstacles with the extended one (i.e., those terms appearing in the summation). We next present a more direct proof to check this point; the result is slightly more general, since we only assume now a sign condition for $\operatorname{Im}\alpha$.

Proof. We carry out the proof in 3D only, since the 2D case can be treated analogously.

(i) Uniqueness. Assuming $u^{in} = 0$, we only need to prove that $u^{sc} = 0$. Note that $u^{sc} = 0$ on ∂D and u^{sc} fulfills the conditions (4.3) as well as the Kupradze's radiation condition.

To prove $u^{sc} \equiv 0$, we need the analogue of Rellich's lemma in linear elasticity (see e.g., [15, Lemma 5.8] and [6]) The Rellich's lemma for the Helmholtz equation can be found in [10, Chapter 2], which ensures uniqueness for solutions to exterior boundary value problems. For $a, b \in R$ such that $a + b = \lambda + \mu$, define the sesquilinear form $\mathcal{E}_{a,b}$ and the traction operator $T_{a,b}$ via

$$\mathcal{E}_{a,b}\,(u, v) := (a+\mu)\sum_{j,k=1}^{3}\frac{\partial u_j}{\partial x_k}\frac{\partial v_j}{\partial x_k} + b\,(\nabla\cdot u)(\nabla\cdot v) - a\operatorname{curl} u\cdot\operatorname{curl} v,$$

$$T_{a,b}\,u := (a+\mu)\frac{\partial u}{\partial \nu} + b\operatorname{div} u\,\nu + a\,\nu\times\operatorname{curl} v,$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. In the generalized Betti's formula (see [27]), we take a special choice of the parameters $a = -\mu$ and $b = \lambda + 2\mu$, so that $a + b = \lambda + \mu$. For notational convenience we write $T = T_{-\mu,\lambda+2\mu}$ and $\mathcal{E} = \mathcal{E}_{-\mu,\lambda+2\mu}$ to indicate the dependance of the traction operator T and the sesquilinear form $\mathcal{E}$ on these parameters. Choose $\epsilon > 0$ sufficiently small and $R > 0$ sufficiently large such that

$$D \subset B_R,\ B_\epsilon(y^{(j)}) \subset B_R\backslash\overline{D}, \quad B_\epsilon(y^{(j)}) \cap B_\epsilon(y^{(m)}) = \emptyset$$

for all $j, m = 1, 2, \cdots, N$ and $j \neq m$. Applying the generalized Betti's formula (see [27]) for u^{sc} to the region $B_{R,\epsilon} = B_R\backslash\overline{D}\backslash\{\overline{\cup_{j=1}^N B_\epsilon(y^{(j)})}\}$, we find

$$\begin{aligned}
0 &= -\int_{B_{R,\epsilon}} (\Delta u^{sc} + \omega^2\, u^{sc})\,\overline{u^{sc}}\,dx \\
&= \int_{B_{R,\epsilon}} \mathcal{E}(u^{sc}, \overline{u^{sc}})dx - \int_{\partial B_{R,\epsilon}} Tu^{sc}\cdot\overline{u^{sc}}\,ds \\
&= \int_{B_{R,\epsilon}} \mathcal{E}(u^{sc}, \overline{u^{sc}})dx - \int_{|x|=R} Tu^{sc}\cdot\overline{u^{sc}}\,ds + \sum_{j=1}^{N}\int_{\partial B_\epsilon(y^{(j)})} Tu^{sc}\cdot\overline{u^{sc}}\,ds, \qquad (4.9)
\end{aligned}$$

where the normal directions at $\partial B_\epsilon(y^{(j)})$ or $\partial B_{R,\epsilon}$ are assumed to point outward. Here we have used the vanishing of u^{sc} on ∂D. Next we estimate the integral on $\partial B_\epsilon(y^{(j)})$ in (4.9) by using the impedance-type boundary condition (4.3). Setting $C_j := (\tau_1 u^{sc})_j \in \mathbb{C}^3$, we derive from (4.4) and (2.42) that

$$u^{sc}(x) = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\frac{C_j}{|x-y^{(j)}|} + \frac{\lambda+3\mu}{8\pi\mu(\lambda+\mu)}\frac{(x-y^{(j)})\otimes(x-y^{(j)})}{|x-y^{(j)}|^3}\cdot C_j + \alpha_j C_j + o(1)$$

as $x \to y^{(j)}$. Let $F(x) = (x\otimes x)\cdot C_j/|x|^3$. Straightforward calculations show that

$$\operatorname{div} F(x) = \frac{C_j\cdot\hat{x}}{|x|^2}, \quad \operatorname{curl} F(x) = \frac{C_j\times\hat{x}}{|x|^2}$$

where $\hat{x} = x/|x| \in \mathbb{S}$. Making use the previous relation, one can calculate for $x \in \partial B_\epsilon(y^{(j)})$ that

$$(\lambda + 2\mu)\mathrm{div}\, u^{sc}\, \nu = -\frac{\nu(\nu \cdot C_j)}{4\pi\epsilon^2} + O(1),$$
$$\nu \times \mathrm{curl}\, u^{sc} = -\frac{\nu \times (C_j \times \nu)}{4\pi\epsilon^2} + O(1),$$

as $\epsilon \to 0$, where $\nu(x) = (x - y^{(j)})/\epsilon$ on $\partial B_\epsilon(y^{(j)})$. By definition of the traction operator, it then follows that

$$Tu^{sc} \cdot \overline{u}^{sc} = -\frac{-1}{16\pi^2\epsilon^3\mu}|C_j|^2 - \frac{\overline{\alpha_j}}{4\pi\epsilon^2}|C_j|^2 + O(\frac{1}{\epsilon}) \quad \text{on} \quad \partial B_\epsilon(y^{(j)})$$

as $\epsilon \to 0$. Since $\mathrm{Im}\,\alpha_j \leq 0$, we get via the mean value theorem that

$$\lim_{\epsilon\to 0} \mathrm{Im} \left(\int_{\partial B_\epsilon(y^{(j)})} Tu^{sc} \cdot \overline{u^{sc}}\, ds \right) = \frac{\mathrm{Im}\,\alpha_j}{4\pi\epsilon^2}|C_j|^2 \leq 0.$$

Now, taking the imaginary part of (4.9) and letting ϵ tend to zero yield

$$\mathrm{Im} \left(\int_{\Gamma_R} Tu^{sc} \cdot \overline{u^{sc}}\, ds \right) \leq 0.$$

Applying [15, Lemma 5.8] gives $u^{sc} = 0$, which proves uniqueness.

(ii) Existence. The solution (4.8) obviously satisfies the Navier equation in $\mathbb{R}^3 \backslash \overline{\Omega}$ and the Dirichlet boundary condition on ∂D. Moreover, $u^{sc} = u - u^{in}$ fulfills the Kupradze radiation condition, because both u_D^{sc} and Γ_D are radiating solutions. Hence, it suffices to check the Impedance-type boundary condition in (4.3) imposed on $y^{(k)}$, $k = 1, \cdots, N$. From the definition of τ_1 and τ_2, follows

$$(\tau_2 \Gamma^D(x,z))_k - \alpha_k\, (\tau_1 \Gamma^D(x,z))_k = \begin{cases} \Gamma^D(y^{(j)}, z), & \text{if } z \in \mathbb{R}^3 \backslash Y\,, \\ \Gamma^D(y^{(j)}, y^{(m)}), & \text{if } z = y^{(m)} \in Y\,,\ m \neq k\,, \\ \chi_{\omega^2} - \alpha_k, & \text{if } z = y^{(k)} \in Y\,, \end{cases}$$

where, here and in the following, we hide the space-indexes and the corresponding sums. This leads to

$$(\tau_2 \Gamma(x, y^{(m)}))_k - \alpha_k\, (\tau_1 \Gamma(x, y^{(m)}))_k = -\left(\Lambda_{\omega^2}^{\alpha,D}\right)^{-1}_{m,k}, \qquad m,k = 1, \cdots, N.$$

On the other hand, since u^{in} is of C^∞-smoothness at $y^{(k)}$, it holds that

$$(\tau_1 u^{in})_k = 0, \quad (\tau_1 u^{in})_k = u^{in}(y^{(k)}).$$

Consequently, by direct calculation we have for $k = 1, \cdots, N$ that

$$\begin{aligned}
&(\tau_2 u)_k - \alpha_k\, (\tau_1 u)_k \\
&= (\tau_2 u^{in})_k - \alpha_k (\tau_1 u^{in})_k + (\tau_2 u^{sc})_k - \alpha_k (\tau_1 u^{sc})_k \\
&= u^{in}(y^{(k)}) + \textstyle\sum_{m,l=1}^N \left(\Lambda_{\omega^2}^{\alpha,D}\right)_{m,l} \left[(\tau_2(\Gamma(\cdot, y^{(m)}))_k - \alpha_k(\tau_1(\Gamma(\cdot, y^{(m)}))_k \right] u^{in}(y^{(l)}) \\
&= u^{in}(y^{(k)}) - \textstyle\sum_{l=1}^N \left[\sum_{m=1}^N \left(\Lambda_{\omega^2}^{\alpha,D}\right)_{l,m} \left(\Lambda_{\omega^2}^{\alpha,D}\right)^{-1}_{m,k} \right] u^{in}(y^{(l)}) \\
&= u^{in}(y^{(k)}) - u^{in}(y^{(k)}) = 0\,.
\end{aligned}$$

□

5 Inverse problems

Having established the forward scattering model in Theorem 4.1, we consider in this section the inverse problem of simultaneously recovering the shape of the extended elastic body D and the location of point-like scatterers $y^{(j)} \in \mathbb{R}^n \backslash \overline{D}$, $j = 1, 2, \cdots, N$, in the case of isotropic and local interactions. The number N of the point-like scatterers is always assumed to be finite but unknown. The factorization method [24, 25] by Kirsch will be adapted to such a multi-scale inverse scattering problem by using different type of elastic waves.

5.1 Factorization method

We consider three inverse problems at a fixed frequency as follows:

(i) Recover the shape $\partial\Omega$ of the extended obstacle and the location $\{y^{(j)} : j = 1, 2, \cdots, N\}$ of point-like scatterers from knowledge of the entire far-field pattern over all incident and observation directions, that is, $\{u^\infty(\hat{x}, d) : \hat{x}, d \in \mathbb{S}^{n-1}\}$.

(ii) Recover $\partial\Omega$ and $\{y^{(j)} : j = 1, 2, \cdots, N\}$ from P-part of the far-field pattern over all observation directions excited by incident compressional plane waves with all incident directions.

(iii) Recover $\partial\Omega$ and $\{y^{(j)} : j = 1, 2, \cdots, N\}$ from S-part of the far-field pattern over all observation directions excited by incident shear plane waves with all incident directions.

Evidently, only one type of elastic waves is used in the last two inverse problems, whereas the entire wave is involved in the first problem. Before stating the factorization method we need to define the far-field operator in linear elasticity. For $g(d) \in L^2(\mathbb{S}^{n-1})^n$, $d \in \mathbb{S}^{n-1}$, we have the decomposition $g(d) = g_s(d) + g_p(d)$ where

$$g_p(d) := (g(d) \cdot d)\ d, \quad g_s(d) := \begin{cases} d \times g(d) \times d & \text{if} \quad n = 3; \\ (g(d) \cdot d^\perp)\ d^\perp & \text{if} \quad n = 2. \end{cases} \tag{5.1}$$

Obviously, g_s belongs to the space of transversal vector fields on $\mathbb{S}^{n-1}$ defined as

$$L^2_s(\mathbb{S}^{n-1}) := \{g_s : \mathbb{S}^{N-1} \to \mathbb{C}^n : g_s(d) \cdot d = 0,\ |g_s| \in L^2(\mathbb{S}^{n-1})\} \tag{5.2}$$

while g_p belongs to the space of longitudinal vector fields on $\mathbb{S}^{n-1}$:

$$L^2_p(\mathbb{S}^{n-1}) := \{g_p : \mathbb{S}^{n-1} \to \mathbb{C}^n : g_p(d) \times d = 0 \text{ in } \mathbb{R}^3,\ g_p(d) \cdot d^\perp = 0 \text{ in } \mathbb{R}^2,\ |g_p| \in L^2(\mathbb{S}^{n-1})\}.$$

For $g \in L^2(\mathbb{S}^{n-1})^n$, introduce the incident fields

$$\begin{aligned} v^{in}_g(x) &:= \int_{\mathbb{S}^{n-1}} \left[g_s(d) e^{ik_s x \cdot d} + g_p(d) e^{ik_p x \cdot d}\right] ds(d), \\ v^{in}_{g_s}(x) &:= \int_{\mathbb{S}^{n-1}} \left[g_s(d) e^{ik_s x \cdot d}\right] ds(d), \\ v^{in}_{g_p}(x) &:= \int_{\mathbb{S}^{n-1}} \left[g_p(d) e^{ik_p x \cdot d}\right] ds(d). \end{aligned}$$

Definition 5.1. *Let v^∞_g be the far-field pattern of the incident wave v^{in}_g, and let $v^\infty_{g,p}$ (resp. $v^\infty_{g,s}$) be the longitudinal (resp. transversal) far-field pattern of the incident wave $v^{in}_{g,p}$ ($v^{in}_{g,p}$). The far-field operators F, F_p and F_s are defined by*

$$\begin{aligned} Fg &:= v^\infty_g\,, \quad (L^2(\mathbb{S}^{n-1}))^n \to (L^2(\mathbb{S}^{n-1}))^n, \\ F_s g_s &:= v^\infty_{g,s}\,, \quad (L^2_s(\mathbb{S}^{n-1}))^n \to (L^2_s(\mathbb{S}^{n-1}))^n, \\ F_p g_p &:= v^\infty_{g,p}\,, \quad (L^2_p(\mathbb{S}^{n-1}))^n \to (L^2_p(\mathbb{S}^{n-1}))^n. \end{aligned}$$

Our inverse scattering problems (IP1) and (IP2) can be equivalently stated as the problems of finding $\partial\Omega := \partial D \cup Y$ from the far-field operators F, F_p and F_s. The operators F_p and F_s are related to F as follows:

$$F_p = \Pi_p \, F \, \Pi_p^*, \quad F_s = \Pi_s \, F \, \Pi_s^*,$$

where Π_α: $(L^2(\mathbb{S}^{n-1}))^n \to (L^2_\alpha(\mathbb{S}^{n-1}))^n(\ \alpha = p, s)$ is the orthogonal projector operator defined by

$$\Pi_p g(d) = g_p(d), \quad \Pi_s g(d) = g_s(d), \qquad g \in (L^2(\mathbb{S}^{n-1}))^n.$$

The adjoint operator $\Pi_\alpha^* : (L^2_\alpha(\mathbb{S}^{n-1}))^n \to (L^2(\mathbb{S}^{n-1}))^n$ of Π_α is just the inclusion operator from $(L^2_\alpha(\mathbb{S}^{n-1}))^n$ to $(L^2(\mathbb{S}^{n-1}))^n$. To state the factorization scheme, for $\mathbf{a} \in \mathbb{S}^{n-1}$ and $y \in \mathbb{R}^n$ we denote by $\Pi^\infty_{\mathbf{a},y}$ the far-field pattern of the outgoing function $x \to \Pi(x, y)\mathbf{a}$. Its compressional and shear parts will be denoted by $\Pi^{\infty,p}_{\mathbf{a},y}$ and $\Pi^{\infty,s}_{\mathbf{a},y}$, respectively. Define $\mathcal{F}_\# := |\mathrm{Re}\,\mathcal{F}| + |\mathrm{Im}\,\mathcal{F}|$ for $\mathcal{F} = F$, F_p and F_s, where $\mathrm{Re}\,\mathcal{F} := (F + F^*)/2$ and $\mathrm{Im}\,\mathcal{F} := (F - F^*)/(2i)$.

Given a non-vanishing vector $\mathbf{a} \in \mathbb{S}^{n-1}$ and a sampling point $y \in \mathbb{R}^n$, the far-field pattern of the radiation function $x \to \Gamma_\omega(x, y)\mathbf{a}$ is given by

$$\Gamma^\infty_{\mathbf{a},y}(\hat{x}) = \begin{cases} e^{-ik_s\hat{x}\cdot y}[\hat{x} \times (\mathbf{a} \times \hat{x})] + e^{-ik_p\hat{x}\cdot y}(\hat{x}\cdot\mathbf{a})\hat{x} & \text{if} \quad n = 3, \\ e^{-ik_s\hat{x}\cdot y}(\mathbf{a}\cdot\hat{x}^\perp)\hat{x}^\perp + e^{-ik_p\hat{x}\cdot y}(\hat{x}\cdot\mathbf{a})\hat{x} & \text{if} \quad n = 2, \end{cases}$$

where $\hat{x}^\perp \in \mathbb{S}^{n-1}$ is orthogonal to $\hat{x}$. By the definition of the projection operators Π_p and Π_s,

$$\Pi_s[\Gamma^\infty_{\mathbf{a},y}(\hat{x})] = \begin{cases} e^{-ik_s\hat{x}\cdot y}[\hat{x} \times (\mathbf{a} \times \hat{x})] & \text{if} \quad n = 3, \\ e^{-ik_s\hat{x}\cdot y}(\mathbf{a}\cdot\hat{x}^\perp)\hat{x}^\perp & \text{if} \quad n = 2, \end{cases}$$

$$\Pi_p[\Gamma^\infty_{\mathbf{a},y}(\hat{x})] = e^{-ik_p\hat{x}\cdot y}(\hat{x}\cdot\mathbf{a})\hat{x} \qquad \text{if} \quad n = 2, 3.$$

Following [16, 18], one can prove that the function $\Gamma^\infty_{\mathbf{a},y}(\hat{x})$ (resp. $\Pi_\alpha[\Gamma^\infty_{\mathbf{a},y}(\hat{x})]$, $\alpha = p, s$) belongs to the range of $\mathcal{F}_\#$ (resp. $\mathcal{F}_{\alpha\#}$) if and only if $y \in D \cup Y$. By Picard's theorem (see, e.g., [25, Theorem 4.8], the set $D \cup Y$ can be characterized through the eigensystems of the far-field operators F, F_s and F_p as follows.

Theorem 5.2. *Suppose that ω^2 is not the Dirichlet eigenvalue of $-\Delta^*$ in D, $\omega \notin S_\alpha$ and $Im\,\alpha_j \le 0$ for all $j = 1, 2, \cdots, N$. Then $y \in \Omega = D \cup Y$ if and only if one of the following three criterions holds*

$$W(y) := \left[\sum_{n=1}^\infty \frac{|(g_n, \Gamma^\infty_{\mathbf{a},y})_{L^2(\mathbb{S}^{n-1})}|^2}{\zeta_n}\right]^{-1} > 0, \tag{5.3}$$

$$W_\alpha(y) := \left[\sum_{n=1}^\infty \frac{|(g_n^{(\alpha)}, \Pi_\alpha[\Gamma^\infty_{\mathbf{a},y}])_{L^2(\mathbb{S}^{n-1})}|^2}{\zeta_n^{(\alpha)}}\right]^{-1} > 0, \qquad \alpha = p, s \tag{5.4}$$

where $\{\zeta_n, g_n\}$ (resp. $\{\zeta_n^{(\alpha)}, g_n^{(\alpha)}\}$) is an eigensystem of the operator $F_\#$ (resp. $F_{\alpha\#}$).

Theorem 5.2 can be proved by combining the arguments of [16] for inverse acoustic scattering from multi-scale sound-soft scatterers and [18] where the factorization method using different type of elastic waves was established for imaging an extended rigid body (that is, $Y = \emptyset$). We remark that, based on the solution representation (4.8), the far-field operators F, F_s, F_p can be factorized as follows (cf. [16, 18]):

$$F = GS^*G^*, \quad F_\alpha = (\Pi_\alpha G)S^*(\Pi_\alpha G)^*, \quad \alpha = p, s,$$

where G and S are modified data-pattern and single layer operators. There is no essential difficulties in applying the range identity established in [25]. The assumptions on the frequency ω ensure that the operators F, F_s and F_p are injective with dense range and also give arise to properties of the middle operator required by the range identity.

5.2 Numerical tests in 2D

In this section, we present numerical examples in two dimensions for testing accuracy and validity of the developed inversion schemes. In our numerical tests, the solutions u_D^{sc} and Γ_D^{sc} are generated by means of the boundary integral equation method and we always set $\omega = 8$, $\rho = 1$, $\lambda = 2$, $\mu = 1$. For simplicity, we assume that ω^2 is not the Dirichlet eigenvalue of $-\Delta^*$ and make the ansatz for the scattered fields corresponding to the rigid scatterer D in the form

$$\begin{aligned} u_D^{sc}(x;d) &= \int_{\partial D} \Gamma_{\omega^2}(x,y)\varphi(y;d)ds(y), \\ \Gamma_D^{sc}(x;z,\mathbf{a}) &= \int_{\partial D} \Gamma_{\omega^2}(x,y)\psi(y;z,\mathbf{a})ds(y), \end{aligned}$$

where $u_D^{sc}(x;d)$ and $\Gamma_D^{sc}(x;z,\mathbf{a})$ are excited by a plane wave of direction $d \in \mathbb{S}^1$ and the point source located at $z \in \mathbb{R}^2\backslash\overline{D}$ with the polarization vector $\mathbf{a} \in \mathbb{S}$, respectively. The densities functions $\varphi(y;d)$ and $\psi(y;z,\mathbf{a})$ can be solved by an equivalent boundary integral equation of first kind. Then, for $z = y^{(m)} \in Y$ one can get the far-field pattern of $\Gamma_D^{sc}(x;z,\mathbf{a})$ as

$$\Gamma_{D,\mathbf{a}}^\infty(\hat{x};y^{(m)}) = \int_{\partial D} \left[e^{-ik_s\hat{x}\cdot y}(\psi(y;y^{(m)},\mathbf{a})\cdot\hat{x}^\perp)\hat{x}^\perp + e^{-ik_p\hat{x}\cdot y}(\hat{x}\cdot\psi(y;y^{(m)},\mathbf{a}))\hat{x}\right] ds(y).$$

By (4.8), we get the far-field pattern for scattering of a plane wave from $\Omega = D \cup Y$ as

$$u^\infty(\hat{x};d) = \sum_{m,l=1}^{N} \Gamma_D^\infty(\hat{x};y^{(m)}) \left[\Lambda_{\omega^2}^{\alpha,D}\right]_{m,l} u_D(y^{(l)};d), \quad \hat{x}, d \in \mathbb{S},$$

where $\Gamma_D^\infty = (\Gamma_{D,\mathbf{e}_1}^\infty, \Gamma_{D,\mathbf{e}_2}^\infty) \in \mathbb{R}^{2\times 2}$ with $\mathbf{e}_1 = (0,1)$, $\mathbf{e}_2 = (1,0)$.

Let N (here we set $N = 64$) incident compressional plane waves $u_p^{in} = d_j \exp(ik_p x\cdot d_j)$ or shear plane waves $u_s^{in} = d_j^\perp \exp(ik_s x\cdot d_j)$ be given at equidistantly distributed directions $d_j = (\cos\theta_j, \sin\theta_j)$ with $\theta_j = 2\pi j/N$, $j = 1, 2, \cdots, N$. Denote by $u_p^\infty(\hat{x};d_j)$, $u_s^\infty(\hat{x};d_j)$ the P-part, S-part of the far-field pattern of the scattered field u^{sc} corresponding to Ω and the incident compressional wave, and by $u_p^\infty(\hat{x};d_j^\perp)$, $u_s^\infty(\hat{x};d_j^\perp)$ the counterparts associated with the incident shear wave. The

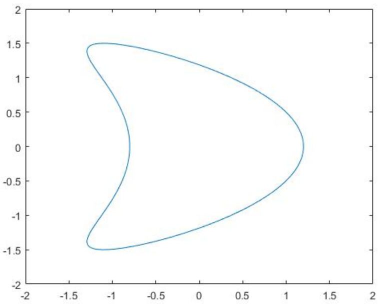


Figure 1: The kite-shaped extended obstacle.

numerical experiments are performed in the following three cases.

PP case: Reconstruct ∂D and Y from $u_p^\infty(d_k;d_j)$ for N incident compressional plane waves $d_j \exp(ik_p x\cdot d_j)$.

SS case: Reconstruct ∂D and Y from $u_s^\infty(d_k;d_j^\perp)$ for N incident shear plane waves $d_j^\perp \exp(ik_s x\cdot d_j)$.

FF case: Reconstruct ∂D and Y from $u^\infty(d_k; d_j)d_k + u^\infty(d_k; d_j^\perp)d_k^\perp$ for N incident elastic plane waves $d_j \exp(ik_p x \cdot d_j) + d_j^\perp \exp(ik_s x \cdot d_j)$.

The measured far-field and near-field data are perturbed by the multiplication of $(1 + \delta\xi)$ with the noise level δ, where ξ is an independent and uniformly distributed random variable generated between -1 and 1. In all of our examples, we suppose that D is a kite-shaped extended obstacle; see Figure 1.

Example 1: (point-scatterers on a line segment) In this example, we compare the reconstruction results for the kite-shaped extended obstacle together with $M = 6$ point-like scatterers on the line segment $2 \times [-2, 2]$ in the SS case, PP case and FF case; see Figures 2 and 3. Here we set $\alpha_j = 0.1, j = 1, \cdots, M$. It can be seen that using the S-part of the far-field pattern still produces satisfactory reconstruction, but the reconstruction from P-part of the far-field pattern is less reliable. This is due the fact that shorter wave length always gives higher resolution and the wave length of shear waves is smaller than that of compressional waves, that is, $\lambda_s < \lambda_p$; see (3.1). In our tests we have $\lambda_s \sim 0.785$ and $\lambda_s \sim 1.57$. In Figure 2, the distance between the point-like scatterers is $l = 0.8$, which is close to the shear wave length but is nearly a half of the compressional wave length. Hence, the point-like scatterers can be distinguished in the SS case rather than the PP case. However, when l is decreased to be 0.4 in Figure 3, using shear waves cannot yield satisfactory reconstructions of the point-like scatterers; cf. (c) and (d) in Figures 2 and 3. It can also be observed that using entire elastic waves would give more reliable reconstructions than the PP and SS cases.

Example 2: (sensitivity to the "impedance" coefficients) In this example, we consider the FF case for reconstructing the same scatterers as in Example 1. We fix $M = 6$, $\mathbf{a} = (1, 0)$. The reconstruction results for different values of α_j are presented in Figure 4. It can be observed that the values of the indicator function around the point-like scatterers grow as the value of α_j decreases, that is, the point-like obstacles are more visible for small α_j. This can be interpreted by the fact that the far-field pattern is proportional to the inverse of α_j; see the solution form (4.8) and the diagonal terms appearing in the matrix (4.5). The same phenomenon was observed in the inverse acoustic scattering problems [16].

Example 3: (random distributed point-like scatterers) In this example, we consider the kite-shaped obstacle and 20 point-like scatterers randomly located in $\{[-3, -2] \cup [2, 3]\} \times [-3, 3]$. The reconstruction from data with 1% noise are shown in Figure 5.

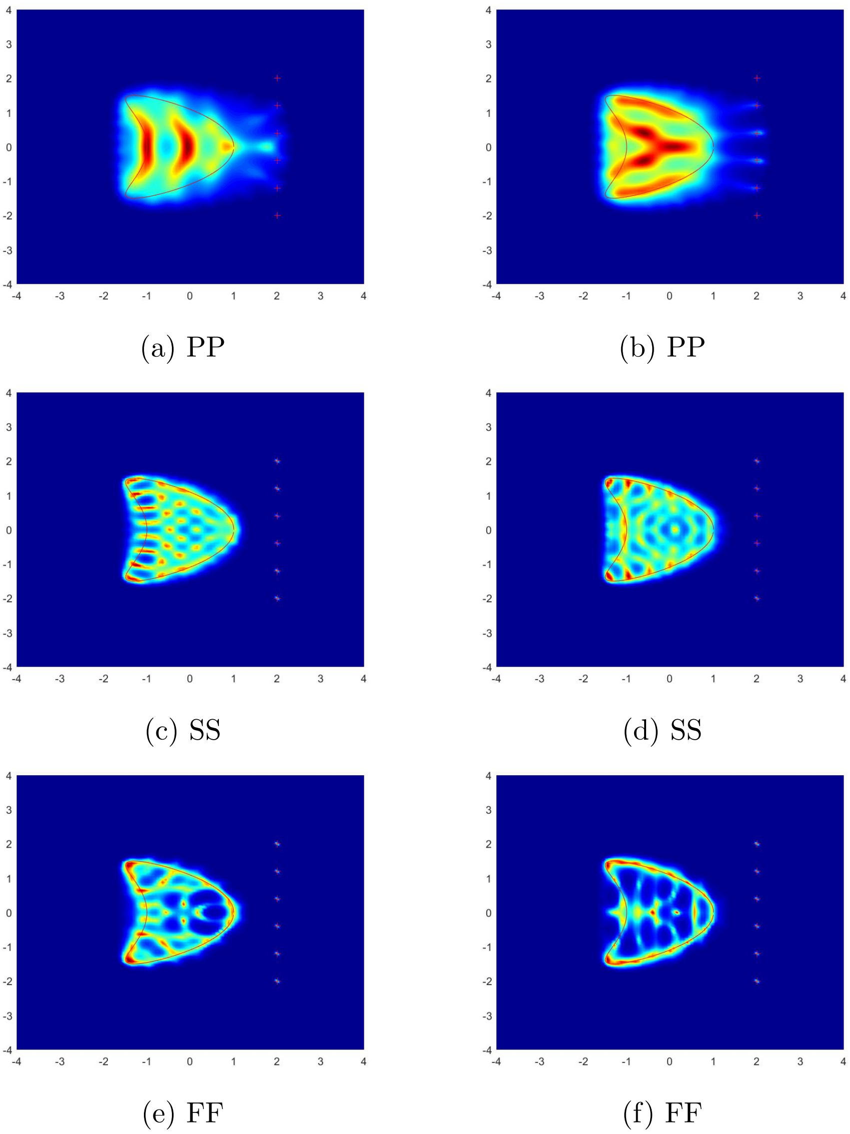
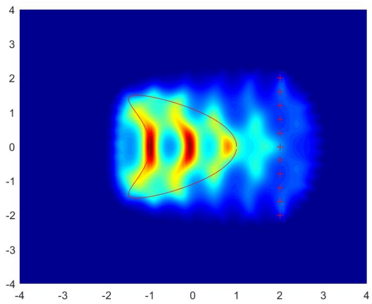
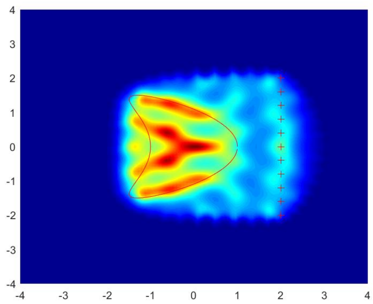


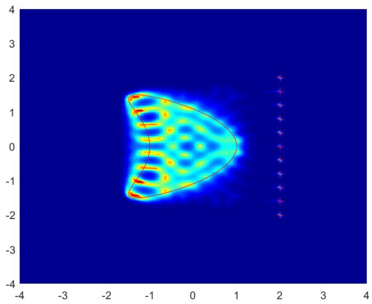
Figure 2: Reconstruction of the kite-shaped obstacle and 6 point-like scatterers for Example 1 with different polarization vectors $\mathbf{a} = (\cos\beta, \sin\beta)$. We set $\beta = 0$ in (a,c,e) and $\beta = \pi/2$ in (b,d,f).



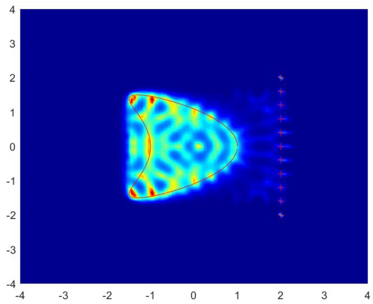
(a) PP



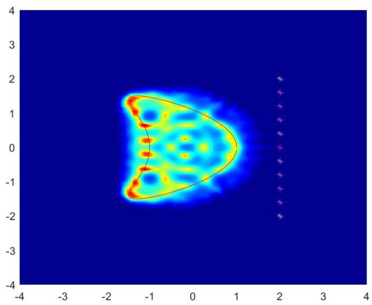
(b) PP



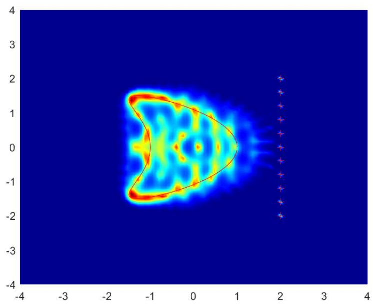
(c) SS



(d) SS

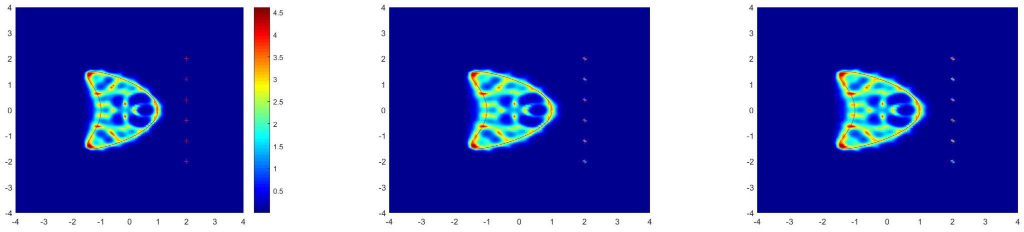


(e) FF

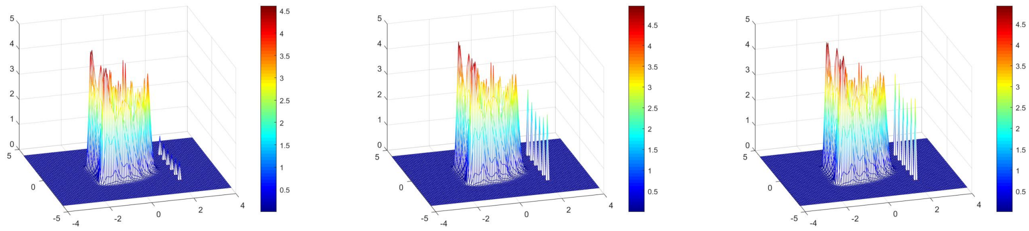


(f) FF

Figure 3: Reconstruction of the kite-shaped obstacle and 11 point-like scatterers for Example 1 with different polarization vectors $\mathbf{a} = (\cos\beta, \sin\beta)$. $\alpha = 0$ in (a,c,e), $\beta = \pi/2$ in (b,d,f).

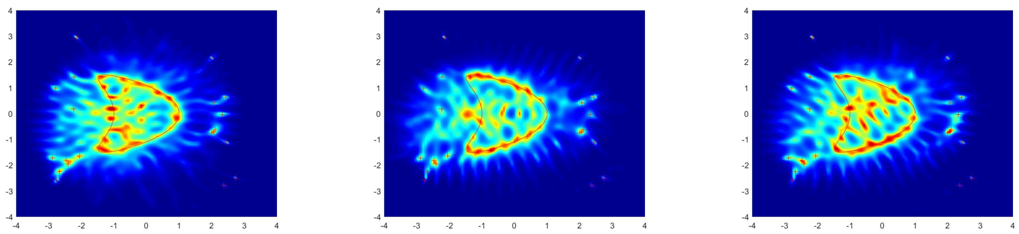


(a) $\alpha_j = 1$ (b) $\alpha_j = 0.1$ (c) $\alpha_j = 0.01$



(d) $\alpha_j = 1$ (e) $\alpha_j = 0.1$ (f) $\alpha_j = 0.01$

Figure 4: Reconstruction of the kite-shaped obstacle and 6 point-like scatterers for Example 2 with different "impedance" coefficients $\alpha_j, j = 1, \cdots, M$..



(a) $\beta = 0$ (b) $\beta = \pi/2$ (c) $\beta = 5\pi/4$

Figure 5: Reconstruction of the kite-shaped obstacle and 20 point-like scatterers for Example 3 with different polarization vectors $\mathbf{a} = (\cos\beta, \sin\beta)$.

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